Cooperative Positioning when using Local Position Information: Theoretical Framework and Error Analysis

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Abstract

Global positioning technologies such as GPS are ubiquitously available for modern smartphones, on-board navigation computers of vehicles, sports appliances, etc. Positioning enables new networking and application solutions, but suffers from insufficient positioning accuracy. We present a way for improving the accuracy of a global positioning system by adding a local positioning system giving relative positions of a group of mobile entities more accurate than the global system communicated among the mobile entities. The improvement is based on transforming the global coordinate system into the local coordinate system and back by translation and rotation operations.

In the paper, we detail the geometric operations and present a novel derivation of how to compute the optimal rotation angle based on covariance measures. Further, we provide estimates for the overall prediction error when using the local positioning system in addition to a global one and a proof of the convergence of the approach. Two major results are the fact that error correlations in the global positioning system increase the prediction error, while error correlations in the local positioning system decrease the overall prediction error.

In a case study, we present results from randomized experiments, as well as from GPS and Differential GPS (DGPS) measurements. Both sets of experiments confirm the theoretical findings.

Index Terms


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1 INTRODUCTION

A key characteristic of mobile systems is their ability to change position while operating. Global positioning systems like GPS, GLONASS, or soon Galileo [1], [2], have revolutionized ship and air technology, and brought a plethora of new applications to mankind, including car navigation, high precision street construction, location based services, finding services and shops in unknown areas, etc. Further, networks make use of location information such as geo-based routing and forwarding protocols [3], [4]. All these technologies suffer from positioning errors, which may range from 1 to 100 m or more. For GPS, for instance, recent trials have reported average errors between 12 m and 24.5 m, depending on factors such as topology [5]. Among the several reasons for errors are further varying signal speed in the atmosphere, multi-path fading, satellite geometry, number of visible satellites, etc. [6].

Some applications are error-tolerant, offering robust services with graceful service degradation depending on the current position uncertainty. In emergency situations, for instance, it is often sufficient to know the position of a mobile phone with an accuracy of within 100 m. For this, GPS or even positioning by using cellular network signaling data is accurate enough. On the other extreme, construction work might depend on highly accurate position data, with an error limit of 10 cm or less. Here, usual positioning technology is not sufficient.

With the introduction of mobile ad-hoc networks (MANETs), e.g., made possible by Bluetooth or WLAN, mobile devices are able to communicate and cooperate with each other on an ad-hoc basis. The idea as presented in this paper is to use a second local positioning technology enabling mobile devices close to each other to calculate local positions independent from the global positioning system, and by sharing this data with each other via an ad-hoc network, achieve higher accuracy. Possible technologies for local positioning are infrared-based, electromagnetic pulse-based, or ultrasound-based. A prototypical implementation based on ultrasound is described in [7] reporting an accuracy of about 8 cm in 90% of the reported cases for local positioning. Though mobile devices would actually spend more energy for communicating with each other, this is a small price to pay compared to the possible gain in positioning accuracy. The mobile nodes can be a group of robots on a mission, but also mobile nodes and devices co-located by chance. Consider for example $N$ cars driving behind each other, each being equipped with a navigation system. Each navigational system may be faced with errors of 10 to 30 m on average, resulting in jumps in the location data, sudden recalculations of the route etc. By using our system and, e.g., a simple distance sensor based on ultra sonic sound or radar, the error may be reduced significantly, thus avoiding the aforementioned problems altogether. We restrict our analysis to the 2D case, since we believe that the by far largest application of such an approach would enhance positioning on the surface of the earth. In
Section 2, related work on the several aspects of cooperative positioning is reviewed.

The major contributions of this paper are the introduction of a theoretical framework for enhancing a global positioning system with a local one and a study of important characteristics of the approach. In Section 3, we present the theory of our proposed scheme, including definitions, how to find the optimal rotation in 2D due to Horn [8] as well as a new derivation based on probability theory. In Section 4, a general error model is introduced as well as an upper bound for the expected error of the rotation angle, and – based on this and additional error analysis – a formula giving upper bounds for the expected and total positioning errors. We further prove convergence and analyze the behavior in the limit. Then, in Section 5, we present results from numerical random experiments and, in Section 6, results from real GPS measurements and show that they behave as expected from the analytical models.

2 Related Work

As of this moment, several additive technologies exist that try to improve the accuracy of global positioning systems. In Assisted GPS, for instance, 3G base stations near the receiver may provide information that speeds up the process of initialization, by sending information about satellite positions and clock times [9]. This way a newly activated GPS receiver may be able to provide location information much faster, and furthermore, with higher accuracy.

A different approach is done by Differential GPS [10]. Here the receiver must be in an area covered by special reference stations whose positions are known precisely. These stations constantly measure their own positions and, knowing their exact position, compute a model of the atmosphere that causes their own positions to be distorted. Receivers in their area then may send their error prone positions to the DGPS provider, e.g., using GPRS or UMTS, who in turn will use the atmospheric model to correct the respective positions, and send them back to the receivers. The errors of DGPS can be in the range of centimeters, but require a constant data connection to a provider offering this service in the visited area. Since DGPS requires a specific service run by an operator, DGPS systems are either run by private consortia or public services, like the Nationwide Differential GPS System (NDGPS).1

The Wide Area Augmentation System (WAAS) and its European counterpart EGNOS as well as the Asian MSAS also rely on reference stations, and provide their differential information by special satellites [11]. They account for ionospheric disturbances and clock drifts, and send correction data from geo-stationary satellites to the terminals.

The idea of cooperative positioning is to accumulate the positioning information that is available locally in distributed nodes, in order to obtain (better) position estimates. Most often

authors propose a hybrid between a global positioning system like GPS, and some way of measuring signal propagation such as, e.g., done in [12], [13]. The cooperative positioning approach is also followed in mobile robot systems, where accumulation errors of the local positioning method occur while the mobile robots try to reach a target position, following, e.g., a dead reckoning approach. Solutions to the error accumulation problem are presented, e.g., in [14]. In contrast to these works, our work does not focus on group movement towards a common target but to increase the positioning accuracy for each participating node.

In the cooperative positioning approaches presented in [15], [16], the authors propose a scheme where scattered nodes use a wireless network to obtain position estimates, here relying on a small number of nodes with known positions. For signal modeling and investigation of accuracy of a local positioning system, in [17], [18], a valuable description is provided. In [19] the authors propose anchor-less positioning, i.e., in an environment without fixed infrastructure. This can be used, for example, in combat situations where base stations are either not available, damaged, and no direct line of sight to satellites is available (such as caves or tunnels). The authors of [20] discuss several approaches of sensor fusion together with fundamental limitations.

Although, in this paper, we also exploit cooperation between mobile nodes, in contrast to the discussed papers, we do not focus on how to create the local positioning system, but assume that such a system is available, and that the local positioning error is way below the one of global positioning systems (see, e.g., the ultrasound-based system prototype ranges below 10cm as described in [7]). Based on these assumptions, we describe how the global position estimates may be corrected based on rotations and translations. Finding optimal rotations in space is an important problem which has been receiving much attention, e.g., in biology or crystallography. In a classic paper, Horn [8] derived expressions for the optimal rotation in 2D and 3D, for 3D using quaternions for representing the rotation in space. A variant of this approach using polynomials is presented in [21] and further simplified in [22]. The authors of [23] generalize to anisotropic error and provide an algorithm for optimal rotations in this situation.

In our paper, we first review the approach described in [8], and then extend related work by presenting a novel calculation of the optimal rotation based on probability theory. We use these findings for improving the accuracy of global positions on a plane by using a local positioning system with better accuracy. We furthermore derive limits for the errors which can be calculated from the measurement data. These results can then be used to derive upper limits of the expected error of our approach, such that this very error can be also estimated from the measurement data.
3 COOPERATIVE POSITIONING

The main idea of our cooperative positioning approach is based on a transformation of the global positions to the local positions to align the geometric structures given by the global and local positions of the mobile entities under consideration. By assuming a more accurate local positioning system, the global positions can be adjusted to the local positions and transformed back into the global coordinate system.

First, we introduce definitions and detail the approach including a description on how to find the optimal affine transformation by optimizing the rotation angle. Then, we introduce a scheme to compute the rotation angle by using probability theory which allows for deriving the error estimates and bounds of this angle, which influence the achievable accuracy of our scheme. Finally, the section concludes with a complexity analysis of our approach.

3.1 Definitions

Assume that there are $N$ nodes in a plane, and each node is localized by a technology $T_1$ for global positioning, which exhibits a significant positioning error, like GPS. Instead of using 2-dimensional vectors for representing the node positions, we use complex numbers, which allow rotations to be represented by a simple multiplication between numbers [24]. Thus, when using $T_1$ let the exact position of a node $j$ be represented by the complex number $z_j = x_j + iy_j$. Since $T_1$ exhibits errors, the position that is actually measured is

$$\tilde{z}_j = z_j + \epsilon_j,$$

with unknown errors $\epsilon_j = \epsilon_{jx} + i\epsilon_{jy}$. Furthermore let $z_c = 1/N \sum_{j=1}^{N} z_j$ be the barycenter of the $N$ exact positions, $\tilde{z}_c = 1/N \sum_{j=1}^{N} \tilde{z}_j$ the barycenter of the measured positions, and $\epsilon_c = 1/N \sum_{j=1}^{N} \epsilon_j$ the average error, yielding

$$\tilde{z}_c = \frac{1}{N} \sum_{j=1}^{N} \tilde{z}_j = \frac{1}{N} \sum_{j=1}^{N} (z_j + \epsilon_j) = z_c + \epsilon_c.$$  

Now assume a second technology $T_2$ yielding only local positions, i.e., the positions of the nodes in relation to a local reference frame. In this local reference frame, let $w_j = u_j + iv_j$ be the exact position of node $i$, and

$$\tilde{w}_j = w_j + \delta_j,$$

the measured positions, with errors $\delta_j = \delta_{jx} + i\delta_{jy}$. Again $w_c = 1/N \sum_{j=1}^{N} w_j$ and $\tilde{w}_c = 1/N \sum_{j=1}^{N} \tilde{w}_j$ denote the barycenters of the exact and measured local positions, and $\delta_c$ the average error. Again we derive

$$\tilde{w}_c = \frac{1}{N} \sum_{j=1}^{N} \tilde{w}_j = w_c + \delta_c.$$
Since both systems denote the positions of the same nodes, we assume that there is a transformation between the global reference frame to the local one. More specifically, we assume that the transformation includes a translation, a rotation, and again a translation. This is due to the fact that, as shall be shown later, for finding an optimal rotation for two point sets, both point sets must first be centered around their centers of mass, i.e., the whole operation involves two translations (centering both point sets) and then the optimal rotation. Therefore, the following relation holds:

\[
\mathbf{w}_j = (\mathbf{z}_j - \mathbf{z}_c)e^{-i\alpha} + \mathbf{w}_c = \mathbf{z}_j e^{-i\alpha} + (-\mathbf{z}_c e^{-i\alpha} + \mathbf{w}_c). \tag{5}
\]

Here \(e^{-i\alpha}\) represents a rotation by some angle \(-\alpha\). Of course, \(\alpha\) itself is not known and must be estimated by using the measured data, yielding the estimate \(\tilde{\alpha} = \alpha + \Delta \alpha\). Note that (6) and therefore (5) is actually an affine transformation, i.e., a multiplication followed by a translation. In our case we restrict the multiplication to a rotation only, without scaling. Now, assuming that the positioning errors of \(T_2\) are much smaller than those from \(T_1\), we undo the transformation (5)

\[
\hat{\mathbf{z}}_j = (\tilde{\mathbf{w}}_j - \tilde{\mathbf{w}}_c) e^{i\tilde{\alpha}} + \tilde{\mathbf{z}}_c = \tilde{\mathbf{w}}_j e^{i\tilde{\alpha}} + (-\tilde{\mathbf{w}}_c e^{i\tilde{\alpha}} + \tilde{\mathbf{z}}_c), \tag{7}
\]

and thus use \(\tilde{\mathbf{w}}_j\) to create estimates \(\hat{\mathbf{z}}_j\) for the global positions \(\mathbf{z}_j\), which hopefully yield better estimates than the measurements \(\tilde{\mathbf{z}}_j\). In the next section, we describe how to find an optimal affine transform (7) for the measured data.

### 3.2 The Optimal Affine Transformation

In this section we shortly review the basics of finding optimal affine transforms of point clouds as described e.g. in [8], [22]. To find the optimal transform of global to local positions, we formalize the transformation as an affine transformation. In general, we assume two sets of measured points \(\tilde{\mathbf{z}}_j\) and \(\tilde{\mathbf{w}}_j\), and ask which affine transform

\[
T(\mathbf{w}) = \mathbf{w} e^{i\tilde{\alpha}} + \mathbf{r}
\]

minimizes the sum of squared errors

\[
E(\mathbf{N}) = \sum_{i=j}^N |\tilde{\mathbf{z}}_j - T(\tilde{\mathbf{w}}_j)|^2 = \sum_{i=j}^N |\tilde{\mathbf{z}}_j - \tilde{\mathbf{w}}_j e^{i\tilde{\alpha}} - \mathbf{r}|^2, \tag{8}
\]

with a rotation by \(\tilde{\alpha}\) and a translation by some \(\mathbf{r} = r_x + ir_y\). In other words, given the measurements \(\tilde{\mathbf{z}}_j\) and \(\tilde{\mathbf{w}}_j\), we look for the parameters \(\tilde{\alpha}, r_x,\) and \(r_y\) that minimize (8). We first
find the translation parameters \( r_x \) and \( r_y \) by setting \( \tilde{\alpha} \) fixed, and setting the partial derivatives of (8) to zero. Therefore we define \( \tilde{\alpha}_j = \tilde{z}_j - \tilde{w}_je^{i\tilde{\alpha}} \), and immediately derive

\[
|\tilde{z}_j - \tilde{w}_je^{i\tilde{\alpha}} - r|^2 = |\tilde{a}_j - r|^2 = (\tilde{a}_j - r)(\tilde{a}_j - \bar{r}) = |\tilde{a}_j|^2 + |r|^2 - 2\text{Re}(\tilde{a}_j\bar{r}).
\]  

(9)

Here, \( \bar{r} \) denotes the complex conjugate of \( r \), and the function \( \text{Re}(\cdot) \) yields the real part of a complex number. When using the notation \( \tilde{a}_j = \tilde{a}_{jx} + i\tilde{a}_{jy} \) we note that \( \text{Re}(\tilde{a}_j\bar{r}) = \tilde{a}_{jx}r_x + \tilde{a}_{jy}r_y \) and \( |r|^2 = r_x^2 + r_y^2 \). Thus

\[
\frac{\partial|r|^2}{\partial r_x} = 2r_x, \quad \frac{\partial|r|^2}{\partial r_y} = 2r_y, \quad \frac{\partial\text{Re}(\tilde{a}_j\bar{r})}{\partial r_x} = \tilde{a}_{jx}, \quad \text{and} \quad \frac{\partial\text{Re}(\tilde{a}_j\bar{r})}{\partial r_y} = \tilde{a}_{jy}.
\]

Summing over \( j \), the partial derivatives are

\[
\frac{\partial E(N)}{\partial r_x} = \sum_{j=1}^{N} (2r_x - 2\tilde{a}_{jx}) = 2Nr_x - 2\sum_{j=1}^{N} \tilde{a}_{jx},
\]

and

\[
\frac{\partial E(N)}{\partial r_y} = \sum_{j=1}^{N} (2r_y - 2\tilde{a}_{jy}) = 2Nr_y - 2\sum_{j=1}^{N} \tilde{a}_{jy}.
\]

Setting them zero yields

\[
r = r_x + ir_y = \frac{1}{N} \sum_{j=1}^{N} \tilde{a}_j = \frac{1}{N} \sum_{j=1}^{N} (\tilde{z}_j - \tilde{w}_je^{i\tilde{\alpha}}) = \tilde{z}_e - \tilde{w}_e e^{i\tilde{\alpha}},
\]  

(10)

i.e., the difference between the two barycenters. Setting (10) into (8) yields

\[
E(N) = \sum_{j=1}^{N} |\tilde{z}_j - \tilde{w}_je^{i\tilde{\alpha}} - (\tilde{z}_e - \tilde{w}_e e^{i\tilde{\alpha}})|^2 = \sum_{j=1}^{N} |(\tilde{z}_j - \tilde{z}_e) - (\tilde{w}_j - \tilde{w}_e)e^{i\tilde{\alpha}}|^2.
\]  

(11)

Eqn. (11) tells us that before solving the problem, we must first subtract the barycenters from the data sets. Now, we define the centered errors of node \( j \) by \( \epsilon_j = \epsilon_{jx} + i\epsilon_{jy} := \epsilon_j - \epsilon_e \) and \( \delta_j = \delta_{jx} + i\delta_{jy} := \delta_j - \delta_e \), and, using (1), (2), (3) and (4), the centered positions by

\[
\tilde{z}_j^{\epsilon} = \tilde{z}_j - \tilde{z}_e = (z_j - z_e) + (\epsilon_j - \epsilon_e) = z_j^{\epsilon} + \epsilon_j,
\]  

(12)

and

\[
\tilde{w}_j^{\epsilon} = \tilde{w}_j - \tilde{w}_e = (w_j - w_e) + (\delta_j - \delta_e) = w_j^{\epsilon} + \delta_j.
\]  

(13)

For finding an angle \( \tilde{\alpha} \) that minimizes (11) we again compute

\[
|\tilde{z}_j^{\epsilon} - \tilde{w}_j^{\epsilon} e^{i\tilde{\alpha}}|^2 = |\tilde{z}_j^{\epsilon}|^2 + |\tilde{w}_j^{\epsilon} e^{i\tilde{\alpha}}|^2 - 2\text{Re}(\tilde{z}_j^{\epsilon} \tilde{w}_j^{\epsilon} e^{i\tilde{\alpha}}) = |\tilde{z}_j^{\epsilon}|^2 + |\tilde{w}_j^{\epsilon}|^2 - 2\text{Re}(\tilde{z}_j^{\epsilon} \tilde{w}_j^{\epsilon} e^{i\tilde{\alpha}}).
\]  

(14)

It follows that in order to minimize (11) we must maximize \( \sum_{j=1}^{N} \text{Re}(\tilde{z}_j^{\epsilon} \tilde{w}_j^{\epsilon} e^{i\tilde{\alpha}}) \). Now,
and therefore we set

\[ f(\tilde{\alpha}) = \sum_{j=1}^{N} \text{re}(\tilde{z}_j \overline{w}_j e^{i\alpha}) = \cos \tilde{\alpha} \sum_{j=1}^{N} (\tilde{x}_j \tilde{u}_j + \tilde{y}_j \tilde{v}_j) + \sin \tilde{\alpha} \sum_{j=1}^{N} (\tilde{y}_j \tilde{u}_j - \tilde{x}_j \tilde{v}_j). \] (15)

To finally calculate the optima, we define \( \tilde{A} = \sum_{j=1}^{N} (\tilde{x}_j \tilde{u}_j + \tilde{y}_j \tilde{v}_j) \) and \( \tilde{B} = \sum_{j=1}^{N} (\tilde{y}_j \tilde{u}_j - \tilde{x}_j \tilde{v}_j) \) and set the derivative of (15) to zero:

\[ f'(\tilde{\alpha}) = -\tilde{A} \sin \tilde{\alpha} + \tilde{B} \cos \tilde{\alpha} = 0, \]

which yields

\[ \tilde{\alpha}_1 = \arctan \frac{\tilde{B}}{\tilde{A}} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \tilde{\alpha}_2 = \pi + \alpha_1, \] (16)

one maximizing, and one minimizing (11). Alternatively, the solutions are given by

\[ \tilde{\alpha}_{1,2} = \arcsin \pm \sqrt{\frac{\tilde{B}^2}{\tilde{A}^2 + \tilde{B}^2}}, \quad \text{or} \quad \tilde{\alpha}_{1,2} = \arccos \pm \sqrt{\frac{-\tilde{B}^2}{\tilde{A}^2 + \tilde{B}^2}}. \]

### 3.3 Probabilistic Derivation

By purely probabilistic means, we provide a novel way for computing the optimal rotation angle of the affine transformation (16). In general, we assume that the node positions follow some distribution [25], and are described by random variables \( X_j, Y_j, U_j, V_j \), with finite expectations

\[ E[X_j], E[Y_j], E[U_j], E[V_j] \]

and variances \( \sigma_X^2, \sigma_Y^2, \sigma_U^2, \sigma_V^2 \). The complex positions are given by

\[ Z_j = X_j + iY_j \quad \text{and} \quad W_j = U_j + iV_j, \] (17)

the sample means by \( X_c, Y_c, U_c, V_c \) (and \( Z_c, W_c \)), and centered positions by

\[ X_c^e = X_j - X_c, \quad Y_c^e = Y_j - Y_c, \quad U_c^e = U_j - U_c, \quad V_c^e = V_j - V_c. \] (18)

Since the sample mean is an unbiased estimator for the expectation, the centered positions have zero expectation, i.e.,

\[ E[X_c^e] = E[Y_c^e] = E[U_c^e] = E[V_c^e] = 0. \]

Furthermore, by definition, there is a fundamental relation between the variables

\[ Z_j^e = X_j^e + iY_j^e = W_j^e e^{i\alpha} = (U_j^e + iV_j^e) e^{i\alpha} = (U_j^e + iV_j^e)(\cos \alpha + i \sin \alpha), \] (19)

i.e.,

\[ X_j^e = U_j^e \cos \alpha - V_j^e \sin \alpha \quad \text{and} \quad Y_j^e = U_j^e \sin \alpha + V_j^e \cos \alpha. \] (20)
Note that from (20) it follows that the $X^c_j$ are not independent from $U^c_j$ and $V^c_j$, and the same is true for the $Y^c_j$. Furthermore, since measuring those values includes some errors, we define the measurement values to be

$$
\tilde{X}^c_j = X^c_j + \epsilon_{jx}, \quad \tilde{Y}^c_j = Y^c_j + \epsilon_{jy}, \quad \tilde{U}^c_j = U^c_j + \delta_{ju}, \quad \text{and} \quad \tilde{V}^c_j = V^c_j + \delta_{ju},
$$

(21)

where the measurement errors $\epsilon_{jx}, \epsilon_{jy}, \delta_{ju}, \text{and} \delta_{ju}$ have zero expectation (thus, the measurements (21) also have zero expectation) and are independent of each other. It follows that

$$
\text{Cov}[\tilde{X}^c_j, \tilde{U}^c_j] = E[\tilde{X}^c_j \tilde{U}^c_j] = E[X^c_j U^c_j] + E[\epsilon_{jx} \delta_{ju}] + E[\epsilon_{jx} \delta_{ju}]
$$

$$
= E[X^c_j U^c_j]
$$

$$
= E[(U^c_j \cos \alpha - V^c_j \sin \alpha) U^c_j]
$$

$$
\text{Cov}[\tilde{Y}^c_j, \tilde{U}^c_j] = E[(U^c_j \sin \alpha + V^c_j \cos \alpha) U^c_j] = \sin \alpha \text{Var}[U^c_j] + \cos \alpha \text{Cov}[V^c_j, U^c_j],
$$

$$
\text{Cov}[\tilde{Y}^c_j, \tilde{V}^c_j] = E[(U^c_j \sin \alpha + V^c_j \cos \alpha) V^c_j] = \sin \alpha \text{Cov}[U^c_j, V^c_j] + \cos \alpha \text{Var}[V^c_j],
$$

(22)

and, similar to above, define

$$
A_{cov} := \text{Cov}[\tilde{X}^c_j, \tilde{U}^c_j] + \text{Cov}[\tilde{Y}^c_j, \tilde{V}^c_j] = \cos \alpha \text{Cov}[U^c_j] + \text{Var}[V^c_j]),
$$

(23)

and

$$
B_{cov} := \text{Cov}[\tilde{Y}^c_j, \tilde{U}^c_j] - \text{Cov}[\tilde{X}^c_j, \tilde{V}^c_j] = \sin \alpha \text{Cov}[U^c_j] + \text{Var}[V^c_j]),
$$

and therefore

$$
\frac{B_{cov}}{A_{cov}} = \frac{\text{Cov}[\tilde{Y}^c_j, \tilde{U}^c_j] - \text{Cov}[\tilde{X}^c_j, \tilde{V}^c_j]}{\text{Cov}[\tilde{X}^c_j, \tilde{U}^c_j] + \text{Cov}[\tilde{Y}^c_j, \tilde{V}^c_j]} = \tan \alpha.
$$

(24)

In case the data has been measured, the respective covariances must be estimated from the sample data yielding an estimate $\hat{\alpha}$. Indeed this result is a new way for finding the optimal rotation angle (16) by purely probabilistic means. Further, this result demonstrates that in order to find the optimal rotation angle, the above defined independence assumptions are important.

### 3.4 Complexity Analysis

The proposed scheme consists of (i) a technology for computing and communicating local positions and (ii) our proposed correction scheme.
For deriving local positions, each node is assumed to repeatedly broadcast a message, which is received by many or all other nodes. By reflecting such messages, the round-trip time between nodes can be used to estimate the distance between any two nodes. Since each node answers to each message, if there are \( N \) nodes, the number of messages would be \( O(N^2) \). A less accurate alternative could also use the signal strength at the receiver as estimator, i.e., nodes would not be forced to answer to each received message. In this case there are only \( O(N) \) messages needed. However, this method would be severely influenced by external factors like shadowing, fading, etc., thus, reducing the feasibility of such an approach significantly.

Computation of the local topology could be either done on one central node, the coordinator, or on all nodes individually. In both cases, an additional communication and computation effort of at least \( O(N^2) \) has to be considered. Once the results from the local positioning system are known, our proposed correction scheme requires only \( O(N) \) for computing and communicating the corrected positions. This can be seen when looking at the necessary computational steps involved to solve equations (2), (4), (12), (13), (25), and (7).

4 Error Analysis

In the following we describe a way for obtaining an error estimate from the measured data.

4.1 Modeling and Estimating Errors

We first define a general model of the errors, which allows to take into consideration error magnitudes as well as correlations. Afterwards, the results of this section are used to derive first error bounds for \( \tilde{\alpha} \), and then for the overall error. These bounds are then used to show convergence. As an important prerequisite, we assume that the used technology yields usable estimates \( \tilde{\epsilon}^2 \approx \text{Var}[\epsilon_{jx}] \), \( \text{Var}[\epsilon_{jy}] \), and \( \tilde{\delta}^2 \approx \text{Var}[\delta_{ju}], \text{Var}[\delta_{ju}] \) and later we will use these and the results of this section to derive further error bounds.

However, until now we have not assumed anything about the nature of the measurement errors. In fact they could be totally independent, normally distributed with zero expectation, which is a popular assumption. Or, they could show high correlations, which might be given by the fact that for instance in GPS, only very few satellites are seen, and their geometry introduces a considerable bias into the positioning errors. For example, although in the long run, errors might have zero expectation, due to geometric bias, for single measurements, all x-values of the observed \( N \) nodes may be highly correlated. Figure 1 shows this situation. On the left the true positions (round dots), as well as the measured positions (crosses) are seen. The arrows denote the x-components of the measurement errors. On the right, these errors are shown, together with their average \( R \), which for this particular measurement is not zero.
However, in the long run, when taking into consideration many such measurements, errors should have zero expectation.

![Diagram](image)

Fig. 1. Left: several sample cases of true positions, measured positions, and errors of the x-component. Right: x-component errors and their average $R$, which is far away from zero for the sample cases.

In order to model random errors that are correlated as explained above, we define the random variable $R$ with $E[R] = 0$ and $\text{Var}[R] = \sigma_R^2 < \infty$, and i.i.d. $G_j$ with $E[G_j] = 0$ and $\text{Var}[G_j] = \sigma_G^2 < \infty$, and split the errors into a common part and a purely random part

$$E_j = R + G_j,$$

where the $G_j$ are assumed to be independent of $R$ and each other. Because of independence, it follows that $E[E_j] = E[R] + E[G_j] = 0$, but on the other hand, once $R$ is known for a single measurement of all nodes, we have $E[E_j | R] = R$. Furthermore

$$\text{Var}[E_j] = E[E_j^2] = \sigma_E^2 = \sigma_R^2 + \sigma_G^2.$$

Two variables $E_j$ and $E_k$ have the correlation

$$\rho = \text{Corr}[E_j, E_k] = \frac{E[(R + G_j)(R + G_k)]}{\sqrt{\sigma_R^2 + \sigma_G^2} \sqrt{\sigma_R^2 + \sigma_G^2}} = \frac{\sigma_R^2}{\sigma_R^2 + \sigma_G^2}, \quad \rho \in [0, 1],$$

which is equivalent to

$$\sigma_R^2 = \rho \sigma_E^2 \quad \text{and} \quad \sigma_G^2 = (1 - \rho) \sigma_E^2.$$

The variance of the sample mean

$$E_c = \frac{1}{N} \sum_{j=1}^{N} E_j = R + \frac{1}{N} \sum_{j=1}^{N} G_j$$

is then given by

$$\text{Var}[E_c] = \frac{\sigma_G^2}{N} + \frac{\sigma_R^2}{N} = \frac{(1 - \rho) \sigma_E^2}{N} + \rho \sigma_E^2.$$

Furthermore, we define the centered errors to be

$$E'_j = E_j - E_c = \frac{N - 1}{N} G_j - \frac{1}{N} \sum_{k=1, k \neq j}^{N} G_k.$$
Their expectations are zero and because of (28)

\[ \text{Var}[E_j^c] = \frac{(N-1)^2}{N^2} \sigma_G^2 + \frac{N-1}{N^2} \sigma_G^2 = \frac{N-1}{N} \sigma_G^2 = (1-\rho) \frac{N-1}{N} \sigma_E^2. \quad (31) \]

The above presented formulas apply for the one-dimensional case. Going back to the complex plane, we assume that the errors \( \epsilon_{jx}, \epsilon_{jy}, \delta_{ju}, \delta_{jv} \) are realizations of random variables \( E_{jx}, E_{jy}, D_{ju}, D_{jv} \) with properties as described above. We further assume that the used technologies yield error estimates \( \tilde{\epsilon}^2 \approx \sigma_E^2 \) and \( \tilde{\delta}^2 \approx \sigma_D^2 \), and estimates \( \tilde{\rho}_\epsilon \) and \( \tilde{\rho}_\delta \) for the respective error correlations. Generalizing from (27) and (29) we then estimate

\[ \mathbb{E} \left[ \left| E_j^c \right|^2 \right] = \mathbb{E} \left[ E_{cx}^2 + E_{cy}^2 \right] \approx 2\tilde{\epsilon}^2 \left( \frac{1-\tilde{\rho}_\epsilon}{N} + \tilde{\rho}_\epsilon \right) < 2\tilde{\epsilon}^2. \quad (32) \]

Estimates for the centered errors are then computed from (31) as

\[ \mathbb{E} \left[ (E_{jx}^c)^2 \right], \mathbb{E} \left[ (E_{jy}^c)^2 \right] \approx \tilde{\epsilon}^2 \left( 1 - \frac{\tilde{\rho}_\epsilon}{N} \right), \quad \text{and} \]

\[ \mathbb{E} \left[ (D_{ju}^c)^2 \right], \mathbb{E} \left[ (D_{jv}^c)^2 \right] \approx \tilde{\delta}^2 \left( 1 - \frac{\tilde{\rho}_\delta}{N} \right). \quad (33) \]

From the latter and (27) we derive

\[ \mathbb{E} \left[ \left| E_j^c \right|^2 \right] \approx 2\tilde{\epsilon}^2 \left( 1 - \frac{\tilde{\rho}_\epsilon}{N} \right) \frac{N-1}{N} < 2\tilde{\epsilon}^2 \]

and

\[ \mathbb{E} \left[ \left| D_j^c \right|^2 \right] \approx 2\tilde{\delta}^2 \left( 1 - \frac{\tilde{\rho}_\delta}{N} \right) \frac{N-1}{N} < 2\tilde{\delta}^2. \quad (35) \]

These intermediate results will later be used in order to describe the positioning error.

4.2 Error Bounds for \( \Delta \alpha \)

As shall be shown later, the positioning error of our proposed scheme depends on the errors of the local and global schemes, but also on the error \( \Delta \alpha \). In this section we provide estimates for \( \Delta \alpha \) that can be computed from the measurement data. Note that there is no guarantee that \( \Delta \alpha \) is actually smaller than these estimates, similar to error estimates produced by positioning systems, which also provide estimates for the expectation of the errors, but are no error bounds themselves. This is why we also carry out experiments and state the probabilities that \( \Delta \alpha \) is actually smaller than these estimates.

In the following we use the \( \tilde{\alpha} = \alpha + \Delta \alpha \) that actually minimizes (11). Using (12) and (13),
we derive
\[
\tilde{A} = \sum_{j=1}^{N} \tilde{x}_j^c \tilde{u}_j^c + \tilde{y}_j^c \tilde{v}_j^c \\
= \sum_{j=1}^{N} (x_j^c + \epsilon_{jx}^c)(u_{ij}^c + \delta_{ju}^c) + (y_j^c + \epsilon_{jy}^c)(v_{ij}^c + \delta_{jv}^c) \\
= \sum_{j=1}^{N} (x_j^c u_{ij}^c + y_j^c v_{ij}^c) + \\
\sum_{j=1}^{N} (x_j^c \delta_{ju}^c + \epsilon_{jx}^c u_{ij}^c + \epsilon_{jy}^c \delta_{jv}^c + y_j^c \delta_{jv}^c + \epsilon_{jy}^c v_{ij}^c + \epsilon_{jx}^c \delta_{jv}^c), \tag{36}
\]
and
\[
\tilde{B} = \sum_{j=1}^{N} \tilde{y}_j^c \tilde{u}_j^c - \tilde{x}_j^c \tilde{v}_j^c \\
= \sum_{j=1}^{N} (y_j^c u_{ij}^c - x_j^c v_{ij}^c) + \\
\sum_{j=1}^{N} (y_j^c \delta_{ju}^c + \epsilon_{jy}^c u_{ij}^c + \epsilon_{jx}^c \delta_{jv}^c - x_j^c \delta_{jv}^c - \epsilon_{jx}^c v_{ij}^c - \epsilon_{jy}^c \delta_{jv}^c). \tag{37}
\]
Now, when setting
\[
A = \sum_{j=1}^{N} x_j^c u_{ij}^c + y_j^c v_{ij}^c, \quad B = \sum_{j=1}^{N} y_j^c u_{ij}^c - x_j^c v_{ij}^c, \tag{39}
\]
\[
e_A = \sum_{j=1}^{N} x_j^c \delta_{ju}^c + \epsilon_{jx}^c u_{ij}^c + \epsilon_{jy}^c \delta_{jv}^c + y_j^c \delta_{jv}^c + \epsilon_{jy}^c v_{ij}^c + \epsilon_{jx}^c \delta_{jv}^c, \tag{40}
\]
and
\[
e_B = \sum_{j=1}^{N} y_j^c \delta_{ju}^c + \epsilon_{jy}^c u_{ij}^c + \epsilon_{jx}^c \delta_{jv}^c - x_j^c \delta_{jv}^c - \epsilon_{jx}^c v_{ij}^c - \epsilon_{jy}^c \delta_{jv}^c. \tag{41}
\]
Further, we note that
\[
\frac{B}{A} = \frac{\tilde{B} - e_B}{A - e_A} \\
= \frac{\tilde{B}}{A} + \frac{e_A \tan \tilde{\alpha} - e_B}{A}. \tag{42}
\]
Now, assume that $\alpha, \tilde{\alpha} \in (-\pi/2, \pi/2)$, and $\alpha, \tilde{\alpha} \neq 0$. At first we look at the case when there is no error, i.e., $\epsilon_j^c = \delta_j^c = 0$. Each point can be rotated back without error, i.e., there is an $\alpha$ that achieves the minimum error $f(\alpha) = 0$. From (16) we know that this minimum is achieved at
\[
\alpha = \arctan \frac{B}{A},
\]
and thus
\[
-\Delta \alpha = \alpha - \tilde{\alpha} = \arctan \frac{B}{A} - \arctan \frac{\tilde{B}}{A}. \tag{43}
\]
Since \( \arctan'(x) = 1/(1 + x^2) \), at least locally for small \( |\Delta \alpha| \) we derive

\[
-\frac{\Delta \alpha}{e_A \tan \alpha - e_B} = \frac{\arctan \frac{\hat{B}}{A} - \arctan \frac{\hat{B}}{A}}{B/A - \hat{B}/\hat{A}} \approx \arctan' \frac{\hat{B}}{A} = \frac{1}{1 + \tan^2 \hat{\alpha}}
\]

and therefore

\[
-\Delta \alpha \approx \frac{1}{1 + \tan^2 \hat{\alpha}} \frac{e_A \tan \alpha - e_B}{A}.
\]

Noting that \( |A| = |\hat{A} - e_A| \geq |\hat{A}| - |e_A| \) (see for example [26]) an optimistic estimate \( L_1 \) for an upper bound on the absolute error \( |\Delta \alpha| \) is therefore given by

\[
L_1 = \frac{1}{1 + \tan^2 \hat{\alpha}} \frac{|e_A| \tan \hat{\alpha} + |e_B|}{|\hat{A}| - |e_A|}.
\]

A less optimistic bound \( L_2 \) can be easily found by using the fact that the difference quotient \( \frac{f(b) - f(a)}{b - a} \) of a function \( f(x) \) being differentiable in \([a,b]\) equals \( f'(\hat{x}) \) for some \( \hat{x} \in [a,b] \), and thus it must also be smaller than the maximum in this interval. Since

\[
\max_{x \in (-\infty, \infty)} \arctan'(x) = 1,
\]

it follows that

\[
\left| \frac{\arctan \frac{\hat{B}}{A} - \arctan \frac{\hat{B}}{A}}{B/A - \hat{B}/\hat{A}} \right| \leq 1.
\]

From the left equation of (44) we get

\[
|\Delta \alpha| \leq \left| \frac{e_A \tan \hat{\alpha} - e_B}{A} \right| \leq \frac{|e_A| \tan \hat{\alpha} + |e_B|}{|\hat{A}| - |e_A|} =: L_2.
\]

Though the two bounds \( L_1 \) and \( L_2 \) look similar they are in fact very different in nature. \( L_2 \) will get much larger for \( |\alpha| \to \pi/2 \), but converge to \( L_1 \) for \( |\alpha| \to 0 \). \( L_1 \) actually gets smaller for \( |\alpha| \to \pi/2 \), but is more useful as an estimate for an upper limit of the average error, rather than an upper bound for all errors.

Now it is clear that in order to use (45) and (46) the exact errors \( e_A \) and \( e_B \) must be known which usually is not the case. However we can at least provide estimates as described in the following. Using the results from Sections 3.3 and 4.1 we define the random variables

\[
F^c_j = X^c_j D^c_{ju} + U^c_j E^c_{jx} + E^c_{jx} D^c_{ju} + Y^c_j D^c_{jv} + V^c_j E^c_{jy} + E^c_{jy} D^c_{jv}
\]

and

\[
H^c_j = Y^c_j D^c_{ju} + U^c_j E^c_{jy} + E^c_{jy} D^c_{ju} - X^c_j D^c_{jv} - V^c_j E^c_{jx} - E^c_{jx} D^c_{jv}.
\]
For computing the variance we use the following Lemma: Let $U, E, D$ be mutually independent (and therefore uncorrelated) variables with zero expectation. Then

$$\text{Var}[UE + ED] = \text{Var}[U]\text{Var}[E] + \text{Var}[E]\text{Var}[D],$$

even though the products share the same variable $E$.

**Proof:**

$$\text{Var}[UE + ED] = \mathbb{E}[(UE + ED)^2] - (\mathbb{E}[UE + ED])^2 = \mathbb{E}[E^2]\mathbb{E}[(U + D)^2] - (\mathbb{E}[E])^2(\mathbb{E}[U] + \mathbb{E}[D])^2 = \text{Var}[E]\text{Var}[U + D] - 0 = \text{Var}[U]\text{Var}[E] + \text{Var}[E]\text{Var}[D].$$

Therefore,

$$\text{Var} \left[ \sum_{j=1}^{N} F_j^c \right] = \sum_{j=1}^{N} \text{Var} \left[ X_j^c D_{ju}^c + U_j^c E_{jx}^c + E_{jx}^c D_{ju}^c \right] + Y_j^c D_{jv}^c + V_j^c E_{jy}^c + E_{jy}^c D_{jv}^c \right] = N \left( \sigma_X^2 \sigma_D^2 + \sigma_U^2 \sigma_E^2 + \sigma_Y^2 \sigma_D^2 + \sigma_D^2 \sigma_E^2 + 2\sigma_E^2 \sigma_D^2 \right) \approx N \left( \tilde{\sigma}_X^2 \tilde{\sigma}_D^2 + \tilde{\sigma}_U^2 \tilde{\sigma}_E^2 + \tilde{\sigma}_Y^2 \tilde{\sigma}_D^2 + \tilde{\sigma}_D^2 \tilde{\sigma}_E^2 + 2\tilde{\sigma}_E^2 \tilde{\sigma}_D^2 \right) =: \tilde{e}_A^2. \quad (47)$$

and

$$\text{Var} \left[ \sum_{j=1}^{N} H_j^c \right] = N \left( \sigma_Y^2 \sigma_D^2 + \sigma_U^2 \sigma_E^2 + \sigma_X^2 \sigma_D^2 + \sigma_Y^2 \sigma_E^2 + 2\sigma_E^2 \sigma_D^2 \right) \approx N \left( \tilde{\sigma}_Y^2 \tilde{\sigma}_D^2 + \tilde{\sigma}_U^2 \tilde{\sigma}_E^2 + \tilde{\sigma}_X^2 \tilde{\sigma}_D^2 + \tilde{\sigma}_Y^2 \tilde{\sigma}_E^2 + 2\tilde{\sigma}_E^2 \tilde{\sigma}_D^2 \right) =: \tilde{e}_B^2. \quad (48)$$

Here, the $\tilde{\sigma}_X^2, \tilde{\sigma}_Y^2, \tilde{\sigma}_U^2, \tilde{\sigma}_Y^2$ can be directly estimated from the measurements. The square roots of these two variances then can be used as estimates $\tilde{e}_A$ and $\tilde{e}_B$ and inserted into (45) and (46) to derive estimates for the optimistic and pessimistic error bounds for $|\Delta\alpha|$.

Figure 2 shows the average $L_1$, $L_2$, and $|\Delta\alpha|$ from creating nodes randomly (using 100 cases for each $N$) and using $\alpha = \pi/3$, $\epsilon = 1$, $\delta = 0.1$. As can be seen, the $L_1$ exhibit much less variation than $|\Delta\alpha|$. Also, both $L_1$ and $L_2$ in the mean indeed provide bounds for $|\Delta\alpha|$, $L_1$ being quite close to $|\Delta\alpha|$, while $L_2$ – as expected – being a much more conservative upper bound.

In order to verify whether the $L_i$ indeed provide upper bounds, or at least good estimates for such, Figure 3 shows the fraction of times the respective $L_1$ and $L_2$ have actually been larger than $|\Delta\alpha|$. As can be seen, $L_2$ reaches 100%, while $L_1$ being true only in 80% of the times.

This does not mean that $L_2$ is in general always true, since it contains estimated parts. Especially for smaller angles $\alpha$, $L_2$ approaches $L_1$ and both show the same reliability. Figure 4
Fig. 2. $L_1$, $L_2$, and $|\Delta \alpha|$ for different settings of $N$, $\alpha = \pi/3$ – averages (thick lines) plotted together with plus/minus one standard deviation (thin lines).

Fig. 3. Probabilities $P[L_1 > |\Delta \alpha|]$ and $P[L_2 > |\Delta \alpha|]$ for different settings of $N$, $\alpha = \pi/3$.

shows the overall probabilities $P[L_1 > |\Delta \alpha|]$ and $P[L_2 > |\Delta \alpha|]$ (averaged of all $N$), for different rotation angles $\alpha = \text{factor} \times \pi$. As can be seen, $L_2$ is mostly an excellent estimator, but has troubles for smaller $\alpha$. $L_1$ is always better than 50%, and mostly around 80%. This result shows that either $L_1$ should be used only for certain estimated $\tilde{\alpha}$, or a weighted sum of both may be used.

4.3 Positioning Errors

In this section, we now use the previous results to determine the errors of the position estimates $\hat{z}_j$ as defined by (7). This means that we write down the error explicitly, and extend the error representation by previously derived and assumed terms. We then present an estimate for the error itself, that can be derived from the measurements and information from the GPS system.
Fig. 4. Probabilities $P[L_1 > |\Delta \alpha|]$ and $P[L_2 > |\Delta \alpha|]$ for different angles $\alpha$. The x-axis shows the \textit{factor} being a multiplicator for $\pi$, i.e. the rotation angle is $\alpha = \text{factor} \times \pi$.

When putting the definitions of $\tilde{\mathbf{w}}_i$, $\tilde{\mathbf{w}}_c$ and $\tilde{\mathbf{z}}_c$ into (7), we derive:

$$
\tilde{\mathbf{z}}_j = (\tilde{\mathbf{w}}_j - \tilde{\mathbf{w}}_c)e^{i\tilde{\alpha}} + \tilde{\mathbf{z}}_c
$$

$$
= ((\mathbf{z}_j - \mathbf{z}_c)e^{-i\alpha} + \delta^c_j)e^{i\tilde{\alpha}} + \mathbf{z}_c + \mathbf{e}_c
$$

$$
= \mathbf{z}_j e^{i\Delta \alpha} + (1 - e^{i\Delta \alpha})\mathbf{z}_c + \mathbf{e}_c + \delta^c_je^{i\tilde{\alpha}}.
$$

(49)

The latter yields the true positioning error of node $j$:

$$
\mathbf{z}_j - \tilde{\mathbf{z}}_j = (1 - e^{i\Delta \alpha})\mathbf{z}_j - (1 - e^{i\Delta \alpha})\mathbf{z}_c - \mathbf{e}_c - \delta^c_je^{i\tilde{\alpha}}
$$

$$
= (1 - e^{i\Delta \alpha})\mathbf{z}_j^c - \mathbf{e}_c - \delta^c_je^{i\tilde{\alpha}}
$$

(50)

Next we note that

$$
|1 - e^{i\Delta \alpha}|^2 = 2(1 - \cos \Delta \alpha) \leq 2(1 - \cos L_{\Delta \alpha}),
$$

(51)

and

$$
|\mathbf{z}_j^c|^2 = |\tilde{\mathbf{z}}_j^c - \mathbf{e}_j^c|^2 = |\tilde{\mathbf{z}}_j^c|^2 + |\mathbf{e}_j^c|^2 - \text{re}[\tilde{\mathbf{z}}_j^c\mathbf{e}_j^c]
$$

(52)

with

$$
\text{re}[\tilde{\mathbf{z}}_j^c\mathbf{e}_j^c] = \tilde{x}_j^c\epsilon_j^c + \tilde{y}_j^c\epsilon_j^c.
$$

(53)

If the $\tilde{x}_j^c, \epsilon_j^c, \tilde{y}_j^c$, and $\epsilon_j^c$ are drawn from i.i.d. random variables

$$
\tilde{\mathbf{Z}}_j^c = \tilde{X}_j^c + i\tilde{Y}_j^c \quad \text{and} \quad \mathbf{E}_j^c = E_{jx}^c + iE_{jy}^c,
$$

as defined in the previous sections (each with zero expectation since they are centered) then
the expectation of (53) is actually zero. Using (52) and (34) the expectation
\[ E \left[ |Z_j|^2 \right] = E \left[ |\tilde{Z}_j|^2 \right] + E \left[ |E_j|^2 \right] \]
\[ \approx \frac{1}{N} \sum_{k=1}^{N} \left( (\tilde{x}_k^c)^2 + (\tilde{y}_k^c)^2 \right) + 2\tilde{c}^2 (1 - \tilde{\rho}_e) \frac{N-1}{N} \]
\[ \approx \tilde{\sigma}_x^2 + \tilde{\sigma}_y^2 + 2\tilde{c}^2 (1 - \tilde{\rho}_e) \frac{N-1}{N}. \] (54)
can be calculated from the measurements. Following (32) and (35), the squared positioning error \( \hat{E}_P^2 \) of the proposed scheme is then defined by
\[ \hat{E}_P^2 = |Z_j - \tilde{Z}_j|^2 \]
\[ = |(1 - e^{i\alpha})|^2 |Z_j|^2 + |E_j|^2 + |D_j|^2 - \]
\[ - \text{re}[ (1 - e^{i\alpha}) Z_j \bar{E}_c ] - \text{re}[ (1 - e^{i\alpha}) Z_j \bar{D}_j e^{-i\tilde{\alpha}} ] + \]
\[ + \text{re}[ E_j \bar{D}_j e^{-i\tilde{\alpha}} ]. \] (55)
By using an argument similar to above the respective interaction parts have zero expectation, and due to (32), (35), (51), (54), and (55) an estimate for the expectation of the squared error is
\[ E \left[ \hat{E}_P^2 \right] \approx 2 \left( 1 - \cos \Delta \alpha \right) \left( \tilde{\sigma}_x^2 + \tilde{\sigma}_y^2 + 2\tilde{c}^2 (1 - \tilde{\rho}_e) \frac{N-1}{N} \right) + \]
\[ + 2\tilde{c}^2 \left( \frac{1 - \tilde{\rho}_e}{N} + \tilde{\rho}_e \right) + 2\tilde{\delta}^2 (1 - \tilde{\rho}_d) \frac{N-1}{N} \] (56)
\[ \leq 2 \left( 1 - \cos L_{\Delta \alpha} \right) \left( \tilde{\sigma}_x^2 + \tilde{\sigma}_y^2 + 2\tilde{c}^2 (1 - \tilde{\rho}_e) \frac{N-1}{N} \right) + \]
\[ + 2\tilde{c}^2 \left( \frac{1 - \tilde{\rho}_e}{N} + \tilde{\rho}_e \right) + 2\tilde{\delta}^2 (1 - \tilde{\rho}_d) \frac{N-1}{N} \] (57)
Equ. (56) shows an estimate for the overall error that contains a term for the global positioning error estimated by \( \tilde{c}^2 \), a term for the local positioning error represented by \( \tilde{\delta}^2 \), and a term representing the error caused by the angle estimate error \( \Delta \alpha \). The latter causes an error that rotates the whole topology, and therefore also the global position variances are magnified by this factor. Unfortunately, Equ. (56) contains the unknown \( \Delta \alpha \) and can therefore not be used for computing an overall error estimate. By using the results from Section 4.2, Equ. (57) gives an estimate for the upper limit of the error, which can be used in practical situations.

Note that the above treatment focuses on the sum of the squared errors. Additionally, also the error distance
\[ |E_j| = \sqrt{E_{jx}^2 + E_{jy}^2} \] (58)
might be of interest. This means that we compute an estimator for
\[ E \left[ \sqrt{E_{jx}^2 + E_{jy}^2} \right]. \] (59)
Note that this cannot be computed just by taking the square root from (56). To see this on an example, consider the case where both \( E_{jx} \) and \( E_{jy} \) are i.i.d. following \( \mathcal{N}(0,σ^2) \). Then

\[
\sqrt{\mathbb{E}[E_{jx}^2 + E_{jy}^2]} = \sqrt{\mathbb{E}[E_{jx}^2] + \mathbb{E}[E_{jy}^2]} = \sqrt{2σ^2} \approx 1.414 σ.
\]

On the other hand, \( \sqrt{E_{jx}^2 + E_{jy}^2} \) follows a two-dimensional Rayleigh(\( σ \)) distribution [25] with expectation

\[
\mathbb{E}\left[\sqrt{E_{jx}^2 + E_{jy}^2}\right] = \sqrt{\frac{π}{2}} σ \approx 1.253 σ.
\]

Thus, for normally distributed errors, the mean error distance (59) is indeed slightly smaller (by a factor of 1.253/1.414 ≈ 0.886) than the square root of the mean squared error (56).

### 4.4 Convergence

In this section, we investigate how the error behaves if the number of nodes rises. Since the estimates are based on erroneous measurements, it is clear that the overall error cannot drop below a certain bound. However, the basic assumption is that more nodes will generally result in better estimates.

We start by showing that in the limit \( N \to \infty \) the angle estimate will be exact, i.e., its error \( Δα \to 0 \). Using the notation from Sections 3.3 and 4.1, and (40) and (41), which determine the error of the angle estimate due to (42), we note that \( e_A, e_B \) and \( A \) given by (39) turn into infinite sums. However, when dividing numerator and denominator of (42) by \( N \), we note

\[
\lim_{N \to \infty} \frac{1}{N} e_A = \text{Cov}[X^c_j, D^{c}_{ju}] + \text{Cov}[U^c_j, D^{c}_{jx}] + \text{Cov}[E^c_{jx}, D^{c}_{ju}] + \text{Cov}[Y^c_j, D^{c}_{jv}] + \text{Cov}[V^c_j, D^{c}_{jy}] + \text{Cov}[E^c_{jy}, D^{c}_{jv}]
\]

\[
= 0
\]

(60)

and likewise

\[
\lim_{N \to \infty} \frac{1}{N} e_B = 0.
\]

Then, by definition of \( A \) and \( B \) from (39) and (24) we use the fact that

\[
\lim_{N \to \infty} |A|/N = |\cos α| \left( \text{Var}[U^c_j] + \text{Var}[V^c_j]\right) > 0.
\]

When using (42) we derive

\[
\lim_{N \to \infty} \frac{\tilde{B}}{A} = \frac{B}{A},
\]

and because of (43) it follows that

\[
A = \arctan \frac{\tilde{B}}{A} - \arctan \frac{B}{A} \to 0
\]

and therefore \( 2(1 - \cos Δα) \to 0 \). Equ. (56) thus converges to

\[
\mathbb{E}\left[\hat{E}_{jh}^2\right] \approx 2\hat{\rho}^2 + 2\hat{\rho}^2(1 - \hat{\rho}).
\]

(61)
Thus, the prediction error (61) in the limit consists of the measurement errors of the global positioning system and the local system. However, if the global errors are uncorrelated, then in the limit this error vanishes, and the resulting error is only determined by the local positioning system. On the other hand, the higher the correlations in the local positioning system, the better, since these errors are multiplied by \((1 - \rho_\delta)\). This can be explained by the fact that for higher correlations of the local errors, local errors tend to overlap, and the common error part simply vanishes when subtracting the center from the local data.

Compared to this, the squared positioning error of node \(j\) when using only measurements \(\tilde{Z}_j\) is

\[
\tilde{E}_P^2 = |Z_j - \tilde{Z}_j|^2 = |E_j|^2 \quad \text{with} \quad \mathbb{E}[\tilde{E}_P^2] \approx 2\tilde{\epsilon}^2. \tag{62}
\]

Thus, in the limit, the proposed scheme on average is better than taking measurements, i.e., \(\mathbb{E}[\tilde{E}_P^2] \leq \mathbb{E}[\tilde{E}_P^2]\), if

\[
2\tilde{\epsilon}^2\tilde{\rho}_\epsilon + 2\tilde{\delta}^2(1 - \tilde{\rho}_\delta) \leq 2\tilde{\epsilon}^2,
\]

which is equivalent to

\[
\tilde{\delta}^2(1 - \tilde{\rho}_\delta) \leq \tilde{\epsilon}^2(1 - \tilde{\rho}_\epsilon). \tag{63}
\]

If the correlations are equal then – not surprisingly – Equ. (63) tells us that the proposed scheme is superior (in the limit) if the error of the local measurements is less than the global one. Otherwise we can see the influence of the correlation. If the correlation \(\tilde{\rho}_\delta\) of the local measurements is large then this has a positive effect. On the other hand, if the correlation \(\tilde{\rho}_\epsilon\) of the global measurements is large then this has a negative effect, because the common error causes more error in the overall position estimate.

5 Theoretical Evaluation

In order to evaluate our theoretical findings, we first perform a study based on simulating an open area as one of the applications of our enhancement scheme. The study is based on a simulation of \(N\) nodes distributed uniformly in a squared area with side length 100 m, i.e., \(\sigma_X^2 = \sigma_Y^2 = 100^2/12\). For each node \(j\) we assume normally distributed measurement errors \(E_j\) for each coordinate, being split into a common part \(R\) and a random part \(G_j\) as given by (26). The global positioning error average magnitude is set to \(\sqrt{\text{Var}[\epsilon_{jx}]} = \sqrt{\text{Var}[\epsilon_{jy}]} = 1\) m, while the local error is \(\sqrt{\text{Var}[\delta_{jx}]} = \sqrt{\text{Var}[\delta_{jy}]} = 10\) cm.

Figure 5 shows the average squared errors (62), i.e., the measurement error of the global positioning system, (55), i.e., the prediction error of our proposed enhancement, and (57) using (45) and (46), i.e., when trying to estimate the errors from the measurement using the two different limits for \(\Delta\alpha\), for uncorrelated errors, i.e., \(\rho_\epsilon = 0\), and \(\alpha = \pi/3\). The thin horizontal
line shows the error limit (61). It can be seen that the results do conform to the theory, i.e., the errors for the proposed scheme converge to the theoretical limit, while the errors of the global positioning system remain constant at $2 = 1^2 + 1^2$ (fluctuations in the figure are due to the fact that the results are estimated from the numerical experiments). Since the errors of the local positioning system are much smaller compared to the global one, their influence is negligible, and the proposed scheme achieves a much better accuracy even when facing high correlations. Due to these small influence, increasing the correlations of the local positioning system in this case does not lead to a substantial reduction of the squared error.

![Nodes on a square](image)

Fig. 5. Errors in a squared area with side length 100 m, average global positioning error 1 m, of the local positioning system 0.1 m, $\rho_e = 0, \rho_\delta = 0, \alpha = \pi/3$. The thin line denotes the error limit.

Figure 6 shows the squared errors with parameters like above, but with correlation $\rho_e = 0.5$. Since the global errors are correlated the estimate for the center of the nodes is distorted and the prediction error rises. However, the resulting error is still much smaller than the measurement error.

Figures 7 and 8 show the same situation, but with a smaller rotation angle $\alpha = \pi/6$. As theory predicts (see Section 4.2) the difference between the estimates $L_1$ and $L_2$ become smaller for smaller $\alpha$, but otherwise we see the same situation as before.

When considering an alternative scenario, where cars are driving on a street behind each other, forming a line, the variance in the direction orthogonal to the driving direction is quite small, while the variance in the driving direction increases with increasing $N$. On the other hand, both $L_1$ and $L_2$ approach zero for increasing $N$, and the result is almost the same as the topology on a square.

For $N$ nodes on a square, Figures 9 and 10 show the probability that the prediction error...
Fig. 6. Errors in a squared area with side length 100 m, average global positioning error 1 m, of the local positioning system 0.1 m, $\rho_\epsilon = 0.5, \rho_\delta = 0, \alpha = \pi/3$.

Fig. 7. Errors in a squared area with side length 100 m, average global positioning error 1 m, of the local positioning system 0.1 m, $\rho_\epsilon = 0, \rho_\delta = 0, \alpha = \pi/6$.

(55) is less than the error estimate (57), for using either $L_1$ or $L_2$ as a bound for $\Delta \alpha$. As expected, using $L_2$ will result in a larger probability, but for smaller $\alpha$, $L_2$ approaches $L_1$ and the difference vanishes. Higher correlations $\rho_\epsilon$ also decrease the probability. Generally more than 70% of the errors are smaller than the bound estimates. Thus, these estimates not only provide a good estimate for the expected errors, but also give a good idea about an upper bound that is not exceeded too often.
Fig. 8. Errors in a squared area with side length 100 m, average global positioning error 1 m, of the local positioning system 0.1 m, $\rho_\epsilon = 0.5$, $\rho_\delta = 0$, $\alpha = \pi/6$.

Fig. 9. Probability that the squared error (55) is less than (57) when using $L_1$, for different settings for $\alpha$ and $\rho_\epsilon$.

6 REAL GPS MEASUREMENTS

We carried out real measurements with nine GPS terminals tracking their position using standard GPS technology. Furthermore, we used a DGPS system that delivers high precision measurements with errors down to a few centimeters, acting as ground truth\(^2\), i.e., by using DGPS we were able to assess the measurement errors of the GPS terminals. The GPS terminals were

\(^2\) [Link: http://www.eposa.at/]
Fig. 10. Probability that the squared error (55) is less than (57) when using $L_2$, for different settings for $\alpha$ and $\rho_\epsilon$.

placed on an open area inside Vienna, without buildings blocking the line of sight to satellites. Note, that the terminals where placed at known distances (with an error in the range of centimeters). These distances are used as the distances an accurate location positioning system would measure. Figure 11(a) depicts the used equipment and Figure 11(b) shows a snapshot taken during the experiments.

We used two types of topologies, the first being a square grid, the second being a straight line. The distance $d$ between neighboring nodes varied between 10 cm up to 10 m. In addition to the erroneous GPS positions, the reference position of each node was also recorded using the DGPS system. The cooperative positioning approach can work with any topology, however
signal run-times might differ.

6.1 Computing 2D Coordinates from Geographical Coordinates

The proposed scheme assumes data in 2D format, which is realistic for a local positioning system. For a global positioning system like GPS, however, data describes positions on the surface of the earth, and is defined by the angles longitude $\lambda$, latitude $\phi$ and elevation $h$. For converting these parameters to a 2D coordinate system we assume that earth is an ellipsoid with a radius at the equator of $a = 6,378,137$ m, and the radius of a longitudinal circle going through the poles as $b = 0.99664719a = 6,356,752$ m.

More elaborate models show that the length $l_{d\phi}(d\phi; \phi)$ of $d\phi$ latitudinal degrees on the surface of the earth at latitude $\phi$ is

$$l_{d\phi}(d\phi; \phi) = (111132.954 - 559.822 \cos(2\phi) + 1.175 \cos(4\phi)) d\phi \approx \gamma_1 d\phi.$$  

The distance $l_{d\lambda,h}(d\lambda, h; \phi)$ representing $d\lambda$ can be calculated by

$$l_{d\lambda}(d\lambda; \phi, h) = (a + h) \frac{\pi}{180} \cos \left( \arctan \left( \frac{b}{a} \tan(\phi) \right) \right) d\lambda = \gamma_2 d\lambda.$$  

For $\phi = 48.24287$ (a point in Vienna) the results are $\gamma_1 = 111195$ (or $\gamma_1 = 111198.6$ if the elevation is also taken into account) and $\gamma_2 = 74276.84$.

The above defined parameters result in a quite good approximation. However since we do have DGPS coordinates at hand, we can also use an approach similar to the proposed scheme to approximate $\gamma_1$ and $\gamma_2$ – both are assumed to be locally constant – from the measurement data. Assume a node $n_j$ is defined by the geographical coordinates $g_j = (\lambda_j, \phi_j)$. First we select one arbitrary node $n_k$ for a fixed $k$ as 2D point of origin. Then we subtract $g_k$ from all other points, resulting in

$$dg_j = (\lambda_j - \lambda_k, \phi_j - \phi_k) = (d\lambda_j, d\phi_j).$$  

The latter defines a local 2D frame of reference with coordinates

$$c_j = (\gamma_2 d\lambda_j, \gamma_1 d\phi_j).$$  

In a second local reference frame for each node $n_j$ we also know its 2D coordinates (which have been manually measured):

$$\tilde{c}_j = (\tilde{x}_j, \tilde{y}_j).$$  

Next we select $N$ pairs of nodes $(n_{j1}, n_{j2})$ and compute their distance $d_j$, which is the same in both local frames of references:

$$(\tilde{x}_{j1} - \tilde{x}_{j2})^2 + (\tilde{y}_{j1} - \tilde{y}_{j2})^2 = d_j^2 = \gamma_2^2(d\lambda_{j1} - d\lambda_{j2})^2 + \gamma_1^2(d\phi_{j1} - d\phi_{j2})^2.$$
This leads to a linear least squares problem with the unknown $x = (\gamma_2^2, \gamma_1^2)^T$, the distance vector $y = (d_1, \ldots, d_N)^T$ and an $N \times 2$ data matrix $X$ made of the squared differences. The solution is

$$x = \begin{pmatrix} \gamma_2^2 \\ \gamma_1^2 \end{pmatrix} = (X^T X)^{-1} X^T y. \tag{66}$$

It is an open question which approach to use, since both yield similar but not equal results. For instance, when using 27 points arranged on one square and 2 lines, the resulting coefficients are $\gamma_1 = 111530.6$ and $\gamma_2 = 73860.4$. In order to validate that these constants yield meaningful results, we took the DGPS data from our real measurements and multiplied the longitude/latitude degrees with $\gamma_2$ and $\gamma_1$. Figure 12 shows the results, and compares them to the local coordinates of the nodes measured in meters. The figure also shows that when optimally rotating the local coordinates to fit the DGPS data, the results are almost identical. Therefore the above constants can be used to transform longitude/latitude into Cartesian coordinates with unit meter.

![Fig. 12. Transforming geographical coordinates into 2D coordinates: DGPS data (transformed), local coordinates, optimal rotation of local coordinates to fit DGPS data.](image)

Either way, once $\gamma_1$ and $\gamma_2$ are estimated for a certain local spot, we can turn geographical coordinates $(\lambda_j, \phi_j)$ into global 2D coordinates by again calculating (64) by choosing any arbitrary point of origin $g_k$ in the area, and using $\gamma_1$ and $\gamma_2$ to calculate (65). After applying our scheme and deriving positions $(\hat{x}_j, \hat{y}_j)$ with higher accuracy, we can accordingly divide by $\gamma_1$ and $\gamma_2$, and add $g_k$ to derive geographical coordinates $(\hat{\lambda}_j, \hat{\phi}_j)$ with higher precision.
6.2 Experimental Results

Using the above date we are able to enhance the GPS positions by using local information. Figure 13(a) shows the situation for points on a line, distance $d$ between two neighbors is 1 m. It can be seen that the error $\Delta \alpha$ is quite large. This is due to the fact that the GPS errors are quite large in relation to $d$, i.e., the variance of the positions of the nodes $\sigma_X$ and $\sigma_Y$. Figure 13(b) shows how $\Delta \alpha$ decreases once the position variances are increased.

Fig. 13. Enhancing geographical positions: points are on a line with (a) $d = 1$ m and (b) $d = 10$ m.

Including both grid and line topologies, Figure 14 shows the average of the error distances (59) as a function of the number of nodes $N$ used. Since when choosing $N$ nodes out of 9, there are $\binom{9}{N}$ such possibilities. We therefore computed the results for all of them and took the average. The figure shows that the error distances indeed decrease with $\sqrt{N}$. However, the results for the grids (G5 and G10) and line L10 show a rather slow decrease. Closely inspecting the errors there, one can see that in these cases, the $y$-components of the errors are highly correlated, prohibiting a further reduction of the error (see Figure 15).

7 Conclusion

In this paper, we present a way for improving the accuracy of a global positioning system for mobile nodes by using a second, local positioning system with better accuracy. The work is based on the assumption that an ad-hoc wireless network is available for exchanging positioning data reliably and with sufficiently low latencies. We provide a novel derivation of how to compute the optimal rotation angle based on the covariance, and furthermore provide estimates
Fig. 14. Mean error (distance) depending on the number of nodes used. For $N = 1$, the average error distance of the original GPS data is shown. Topologies are Grid (G) and Line (L), neighbor distances are $d = 1, 5, 10$ m.

Fig. 15. Correlations of the error components in x- and y-components for Grid (G) and Line (L) settings, $d=1,5,10$.

for the overall prediction error when using the local positioning system in addition to the global one. We also prove convergence of the approach, and prove the fact that error correlations in the global positioning system increases the error, while error correlations in the local positioning system actually decrease the overall prediction error. Furthermore we prove that in the absence of global error correlations, the prediction error in the limit is actually determined by the local positioning system, and can therefore be enhanced significantly, depending on the technology used.

In a case study we present results from randomized experiments. The presented formulas do not assume any distribution, but will work for all distributions with finite variance. We also
present results from real measurements, where we used GPS and DGPS data to evaluate our scheme. Both sets of experiments confirm the presented theory.

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