On finding a minimal enclosing parallelogram (Abstract)

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Let C be a convex polygon in the plane with n vertices. We denote by P_C a parallelogram that encloses C and that has minimal area among all such parallelograms. In this paper, we show how to compute P_C optimally in time $O(n)$ if the ordered list of edges or vertices of C is given. The problem is of interest in digital signal processing.

Related work has been done on finding a minimal enclosing triangle, see e.g. [OAMB86], a minimal enclosing square [FS75], a minimal enclosing k-gon [ACY85], and a minimal enclosing k-gon that has sides of equal lengths or a fixed-angle sequence [DA84]. Note that whereas e.g. a rectangle would be contained in the latter class of polygons, our problem is different since the angles of the desired enclosing parallelogram are not given in advance.

We first establish a result that allows us to restrict our solution space to all pairs of antipodal edge-vertex pairs of the polygon. Since there are $O(n)$ antipodal pairs, checking all possible configurations yields a simple $O(n^2)$ algorithm to compute the optimal enclosing parallelogram. Exploiting more properties of our problem, we show that only a linear number of configurations, i.e. pairs of antipodal edge-vertex pairs, are eligible for the optimal solution, and we give an linear-time algorithm to find it. We also implemented both algorithms in C++ and found that the linear-time algorithm is not substantially more difficult to implement than that of the quadratic algorithm. Furthermore, the linear-time algorithm is faster than the quadratic algorithm for even the smallest problem size of $n = 5$.

The application that motivated this research is compressing two-dimensional signals (e.g. images) based on their frequency content [ETS94]. Specifically, we are interested in designing rational decimation systems that reduce the number of input samples by a rational factor. A rational decimation system extracts a specific portion of the frequency content (the passband) and resamples the resulting signal at its Nyquist rate. Rational decimation systems are realized by a cascade of four linear operators– modulator, upsampler, filter, and downsampler. In the two-dimensional case, the modulation factor n_0, the upsampling matrix L, the filter passband specifications, and the downsampling matrix M can be computed directly from any parallelogram that circumscribes the passband and has vertices which are rational multiples of π. Therefore, in order to optimize the compression ratio of the overall system, we need to find the parallelogram of minimal area that circumscribes the passband.

Our design procedure takes the vertices of the desired passband and returns the decimator system parameters n_0, L, and M. The vertices would be sketched with a mouse, typed in, or defined by mathematical formulas. The design algorithm amounts to (1) snapping the vertices of the desired passband to grid points that are rational multiples of π, (2) finding the convex hull of the rational vertices, (3) computing the minimal enclosing parallelogram, and (4) calculating the design parameters. Figure shows the automatic design

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```
{n0, L, M} =
  DesignDecimationSystem2D[
    sketchedPolyPoints,
    Justification -> All, Mod -> 10 ];
```

The theoretical upper limit on the compression ratio, computed as the ratio of $4\ \pi^2$ over the area of the original polygon, is $\frac{50}{9}$-to-1, which is 5.55-to-1.
Packing efficiency by the parallelogram is 80%.
The compression ratio is 40-to-9 (4.44-to-1).

$$n_0 = \begin{bmatrix} \frac{-3\,\pi}{10} \\ \frac{2\,\pi}{5} \end{bmatrix} \quad L = \begin{bmatrix} 3 & 0 \\ -48 & 3 \end{bmatrix} \quad M = \begin{bmatrix} 5 & -2 \\ -80 & 40 \end{bmatrix}$$

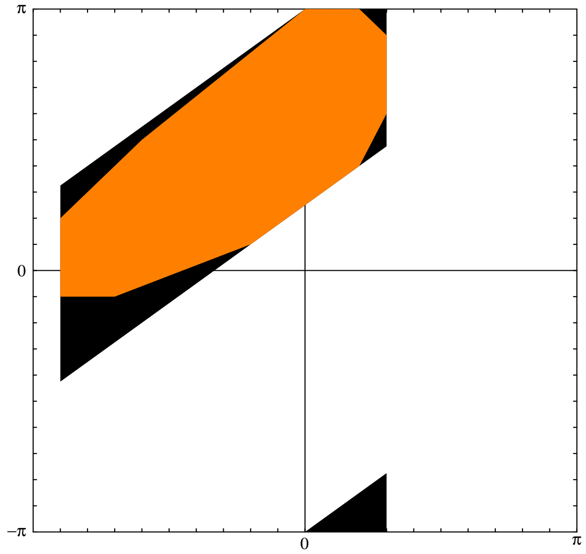

Figure 1: Automatic Design of a Rational Decimation System

of a rational decimation system from a user's sketch of the desired passband based on our implementation in the Mathematica symbolic mathematics environment.

The full details of the algorithms to find the minimal enclosing parallelogram are available as a technical report [STW94]. Details of the digital signal application may be found in [ETS94].

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