Approximating Fault-Tolerant Domination in General Graphs

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Minimum Dominating Set

- Can be approximated with ratio
  - \( \ln(n) - \ln(\ln(n)) + 0.78 \) \([\text{Slavík, 1996}]\)
  - \( H_{\Delta+1} - 0.5 < \ln(\Delta + 1) + 0.5 \) \([\text{Chlebík and Chlebíková, 2008}]\)

- NP-hard lower bound of
  - \( 0.2267 \ln(n) \) \([\text{Alon, Moshkovitz and Safra, 2006}]\)
Minimum $k$-tuple Dominating Set

- $k$-tuple dominating set:
  - Every node should have $k$ dominating nodes in its neighborhood
    
    [Harary and Haynes, 2000 and Haynes, Hedetniemi and Slater, 1998]

- Can be approximated with ratio
  - $\ln(\Delta + 1) + 1$ [Klasing and Laforest, 2004]
Minimum $k$-Dominating Set

- $k$-dominating set:
  - Every node should be in the dominating set or have $k$ dominating nodes in its neighborhood [Fink and Jacobson, 1985]

- Best known approx.-ratio
  \[(e^2 + e)\ln(\Delta)\] [Kuhn, Moscibroda and Wattenhofer, 2006]
Overview of the remaining Talk

- $k$-tuple domination vs $k$-domination

- NP-hard lower bound for $k$-domination

- Improved approximation ratio for $k$-domination
$k$-tuple Domination versus $k$-Domination

- $k$-tuple dominating set only exists if min. degree $\geq k - 1$

- Every $k$-tuple dominating set is a $k$-dominating set

- But how “bad” can a $k$-tuple dom. set be in comparison?
$k$-tuple Domination versus $k$-Domination: With $k = 2$

- $|K| = k = 2$, $|M| = 9$, $\left\lfloor \frac{M}{K} \right\rfloor = \left\lfloor \frac{9}{2} \right\rfloor = 5$
- At least $|M| = 9$ nodes for a $k$-tuple dom. set
- But $|K| + \left\lfloor \frac{M}{K} \right\rfloor = 7$ nodes suffice for a $k$-dom. set
$k$-tuple Domination versus $k$-Domination

- $M \to \infty$: Off by a factor of nearly $k$!

- For $1 < \alpha < k$ and $n \geq k - 1 + \frac{(k-1)^2}{\alpha-1}$: Off by a factor $\geq \frac{k}{\alpha}$ (tight)
NP–hard lower bound for $k$–domination

- NP-hard lower bound for 1-domination
  - $0.2267 \ln(n)$ [Alon, Moshkovitz and Safra, 2006]

- If we could approx. $k$-dom. set with ratio of $s(n)$
  - Then build a $k$-multiplication graph:

\[ G \]

Example for $k = 3$

- NP-hard lower bound for $k$-domination
  - $0.2267/k \ln(n/k)$
Improved approximation ratio for $k$–domination

- Utilizes a **greedy**-algorithm

- Use "**degree**" of $k$–domination per node
  - $k$, if in the $k$-dominating set
  - else **#neighbors** in the $k$-dominating set, but at most $k$

- Pick a node that **improves total sum** of degree the **most**
When does the Greedy Algorithm finish?

- Let a fixed optimal solution have \( r > 1 \) nodes

- Greedy does at least \( 1/r \) of remaining work per step

- If it does more, also good 😊

- Total amount of work is \( n \cdot k \)

- This gives an approximation ratio of roughly \( \ln(n \cdot k) + 1 \)
When to stop when chopping off...

- When is chopping off \( \frac{1}{r} \) of the remaining work ineffective?

- When remaining work is less than \( r \)

- Then **at most** \( r \) more steps are needed

- Stop chopping after \( \ln\left(\frac{nk}{r}\right)/\ln\left(\frac{r}{r-1}\right) \) steps

- Gives an approx. ratio of \( 1 + \ln\left(\frac{nk}{r}\right)/r \cdot \ln\left(\frac{r}{r-1}\right) \)
Calculating the approximation ratio

- $1 + \ln\left(\frac{nk}{r}\right)/r \cdot \ln\left(\frac{r}{r-1}\right)$ does not look too nice...

- 1) $\frac{1}{\ln\left(\frac{r}{r-1}\right)} \leq r \left(1 - \frac{1}{2r}\right) < r$

- 2) $\frac{nk}{\Delta + k} \leq r \iff \frac{nk}{r} \leq \Delta + k$

- Yields: Approx. ratio of less than $\ln(\Delta + k) + 1$

- $\ln(\Delta) + 1.7 < \ln(n) + 1.7$
Extending the Domination Range

• Instead of dominating the 1-neighborhood...

• ... dominate the $h$-neighborhood

• Often called $h$-step domination  cf. [Hage and Harary, 1996]
Extending the Domination Range

• The black nodes form a **2-step** dominating set

• But **not** a 2-step 2-dominating set!
Extending the Domination Range

• Instead of having \( k \) dominating nodes in the \( h \)-neighborhood ...
  – (unless you are in the dominating set)

• ... have \( k \) \textbf{node-disjoint paths} of length at most \( h \)

• Results in approximation ratio of:

• \( \ln(\Delta_h + k) + 1 < \ln(n) + 1.7 \)
Thank you