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On Objective Conflicts and Objective Reduction in Multiobjective Optimization

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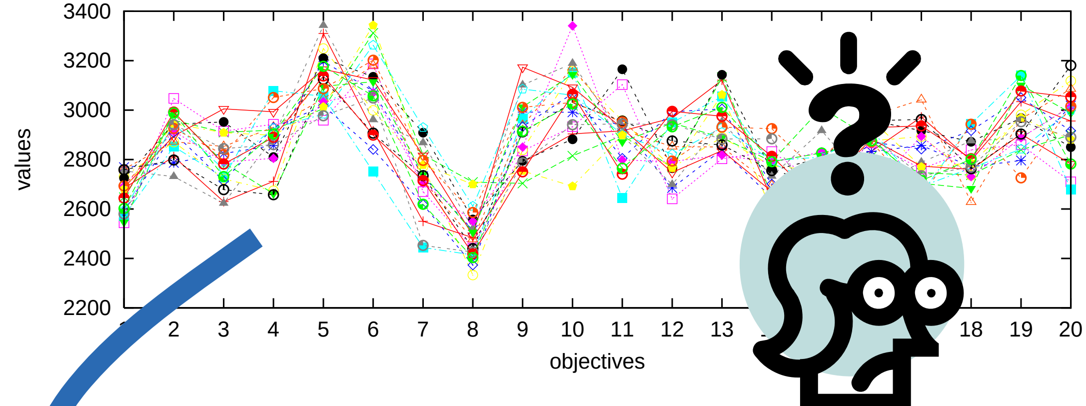
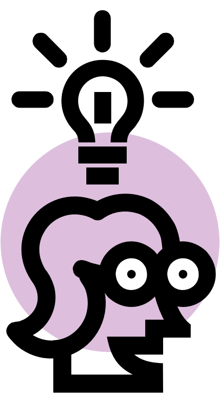
(Approximation of) efficient set

Multiobjective Problem

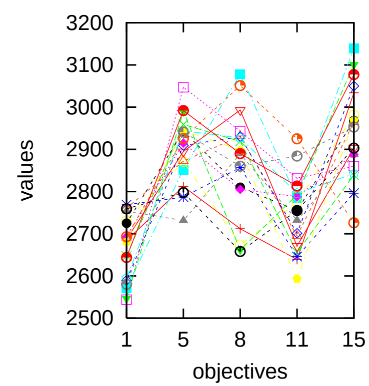
$$\begin{aligned}&\text{min. } \{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\}\\&\text{subject to } \mathbf{x} \in S\\&\text{and } f_i : \mathbb{R}^n \rightarrow \mathbb{R}\end{aligned}$$

Generating method



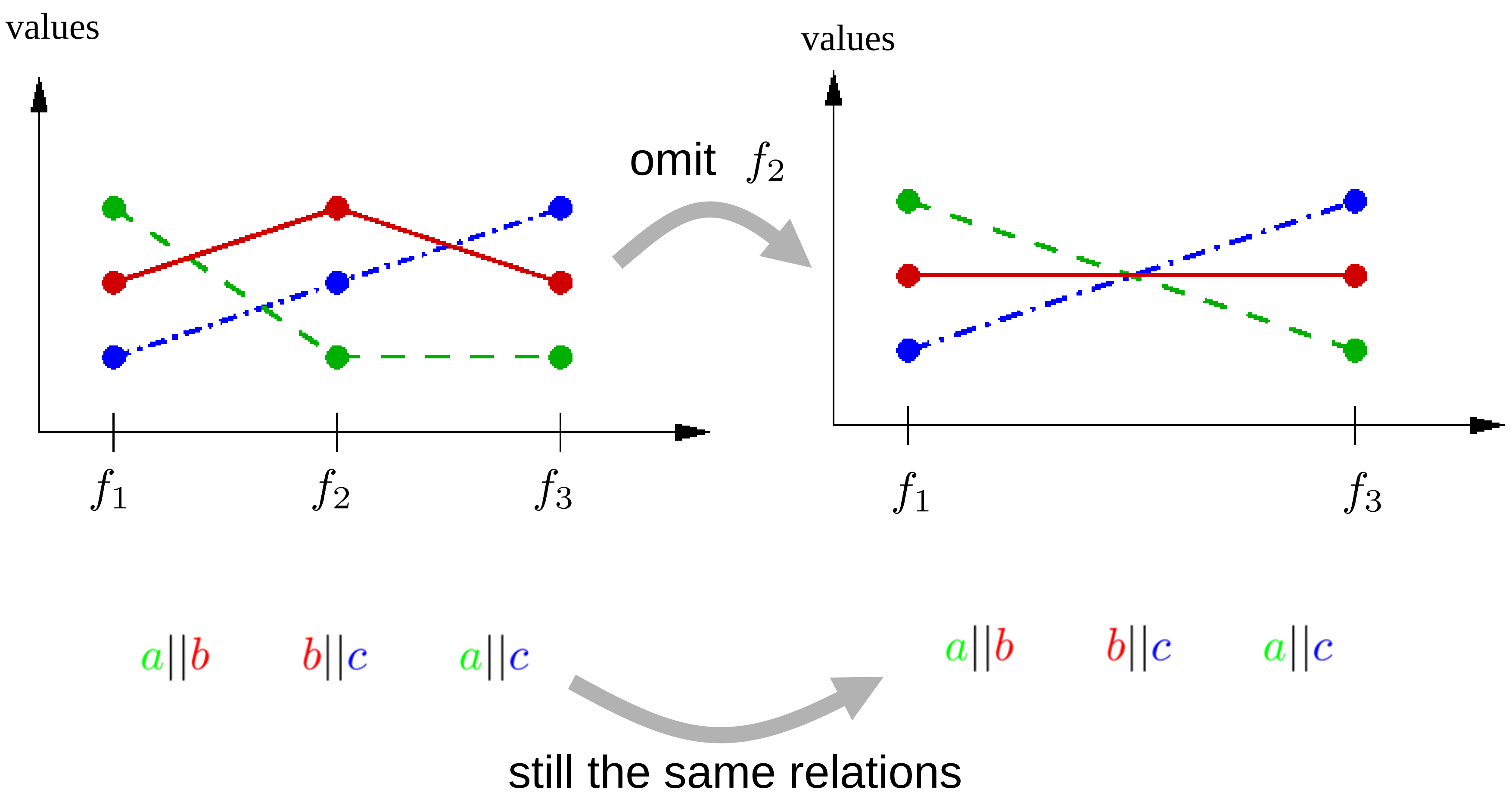


reduce number of objectives



⇒ assist decision maker

Example


values
f_1
f_2
f_3
omit f_2
values
f_1
f_3
$a||b$ $b||c$ $a||c$
$a||b$ $b||c$ $a||c$
still the same relations

Outline

- Objective Reduction in Decision Making Step
  - Objective reduction possible without changing/slightly changing the problem?
  - How to compute a minimum objective set?
- Objective Reduction During Search
  - How can a objective reduction method be used within the search?
  - Is objective reduction suitable in general?
  - What's the problem structure "on the way towards the Pareto front"?

Outline

- Objective Reduction in Decision Making Step
  - Objective reduction possible without changing/slightly changing the problem?
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  - Is objective reduction suitable in general?
  - What's the problem structure "on the way towards the Pareto front"?

Key Questions (Objective Reduction in Decision Making Step)

- Objective reduction possible without changing/slightly changing the problem?
  - How to describe conflicts between objective sets?
- How to compute a minimum objective set?
  - Can we guarantee a lower bound on the error we make?

Related Work

- Omitting redundant objectives:
  - Agrell (1997), Gal and Leberling (1977)
    - Not suitable for black-box optimization
- PCA based objective reduction:
  - Deb and Saxena (2005)
    - Cannot guarantee preservation of dominance structure
- Various conflict definitions:
  - Deb (2001); Tan et al. (2005)
    - conflict as a property of the problem itself
  - Purshouse and Fleming (2003):
    - *objective pairs* conflict if $\geq$ 2 solutions incomparable wrt the objective pair

Open Questions

- Conflicts between arbitrary objective sets
- Objective reduction with
  - preservation of problem structure
  - slight changes in problem structure

  in a black-box scenario
- “Real“ problems

Key Contributions

- Generalization of Objective Conflicts
- The Minimum Objective Subset Problems
  - Exact and heuristic algorithms
- Objective reduction for selected problems

Key Contributions

- **Generalization of Objective Conflicts**
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  - Exact and heuristic algorithms
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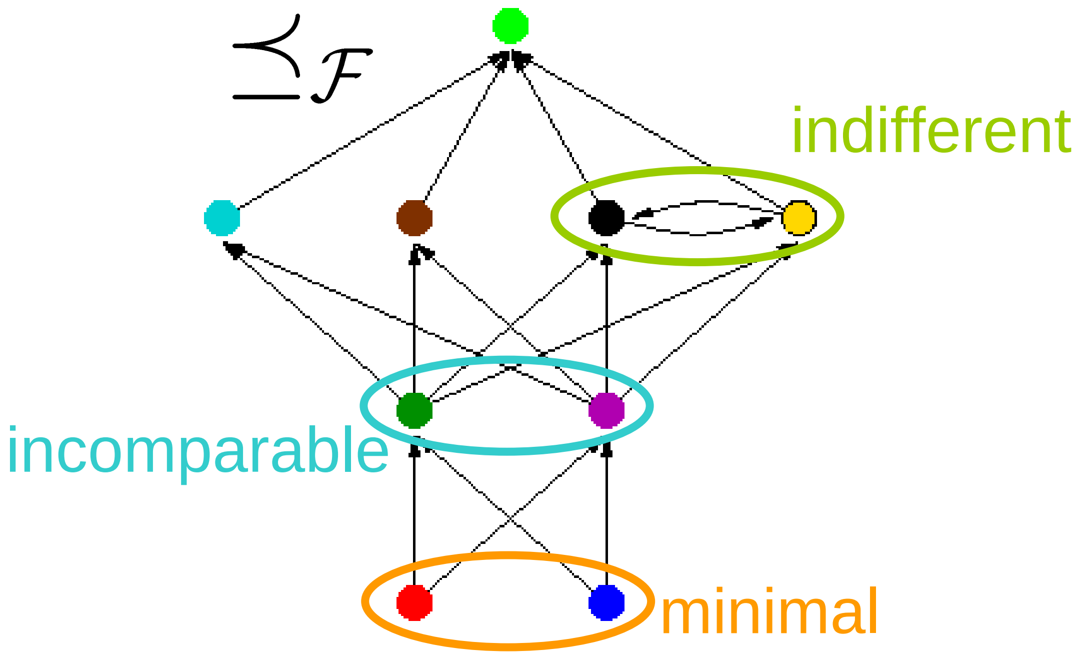
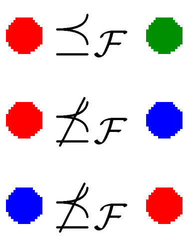
Relation Graphs and Dominance

- For a multiobjective problem, the question is to find the minimal elements of a given (pre)order $(X, \leq)$
- Here, we restrict to the weak dominance relation $\preceq_{\mathcal{F}}$

$$\preceq_{\mathcal{F}} := \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x}, \mathbf{y} \in X \wedge \forall f_i \in \mathcal{F} : f_i(\mathbf{x}) \leq f_i(\mathbf{y})\}$$

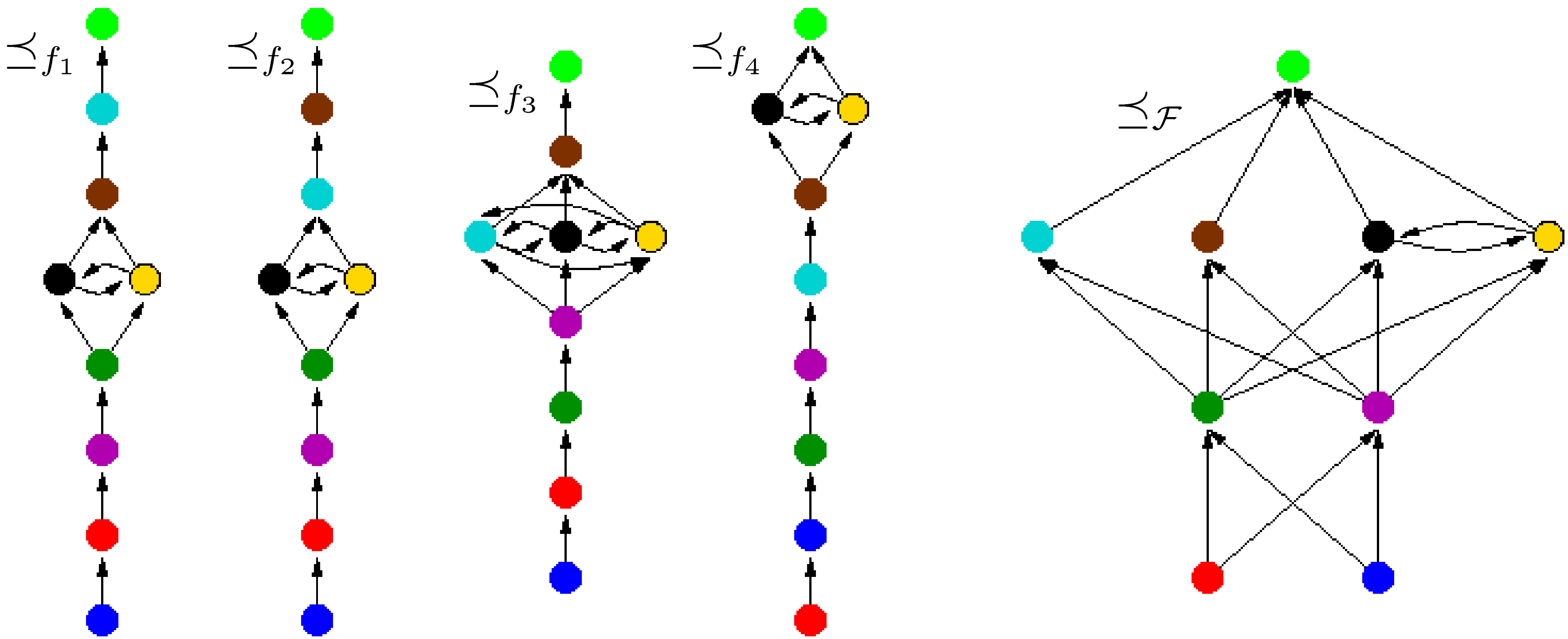




(reflexive and transitive edges are omitted)

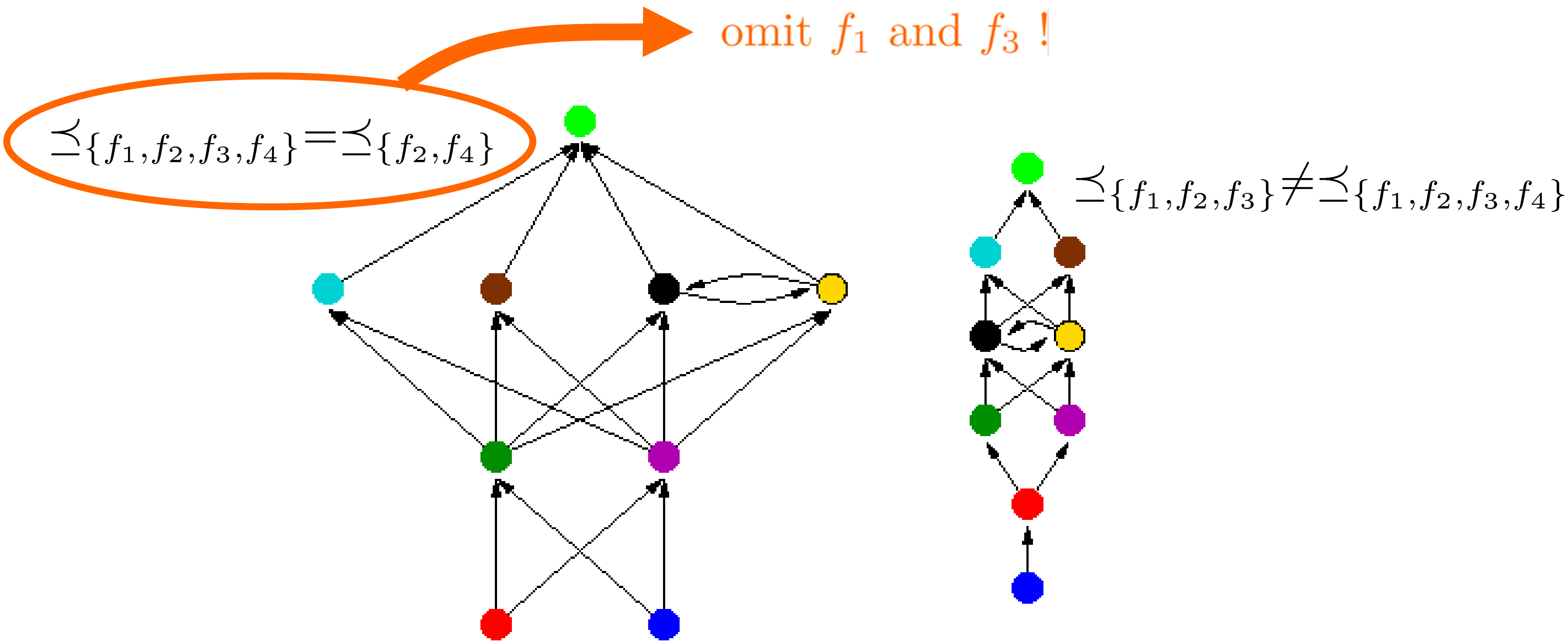
Intersection of Linear (Pre)Orders

- Single objectives induce linear (pre)orders $\preceq_{f_i}$
- Their intersection yields $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$
- Thus, the omission of objectives can only
  - make incomparable solution pairs comparable and
  - comparable solutions indifferent
  - add edges in relation graph



Objective Conflicts

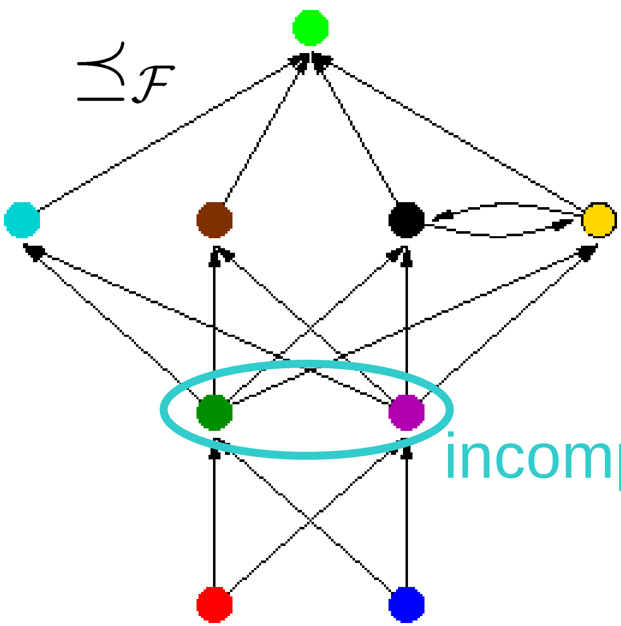
- Objective sets conflict if they induce different relations
  - **Definition:** $\mathcal{F}_1$ nonconflicting with $\mathcal{F}_2$ iff $\preceq_{\mathcal{F}_1} = \preceq_{\mathcal{F}_2}$
  - Omit objectives in $\mathcal{F} \setminus \mathcal{F}'$ if $\mathcal{F}' \subseteq \mathcal{F}$ is nonconflicting with $\mathcal{F}$ and preserve the dominance structure

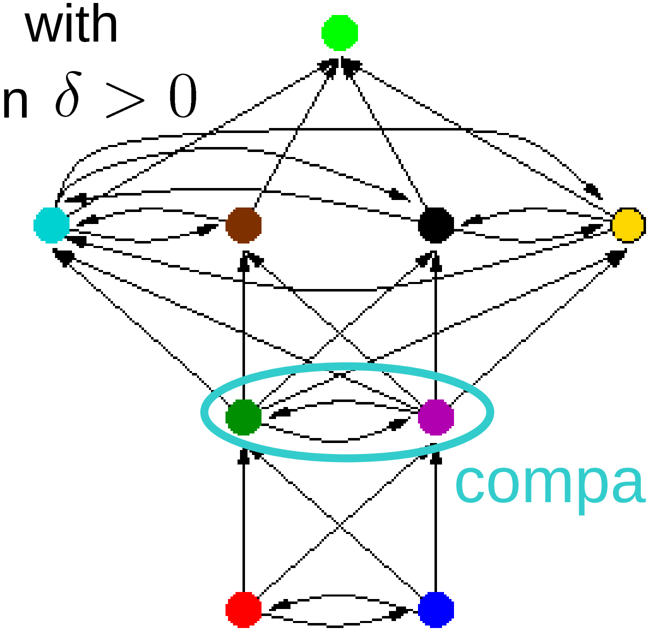


Generalization of Objective Conflicts

- Sometimes the limitation of preserving the problem structure is too strict
- Generalization to δ-conflict based on ε -dominance relation needed (now, objective values are used)

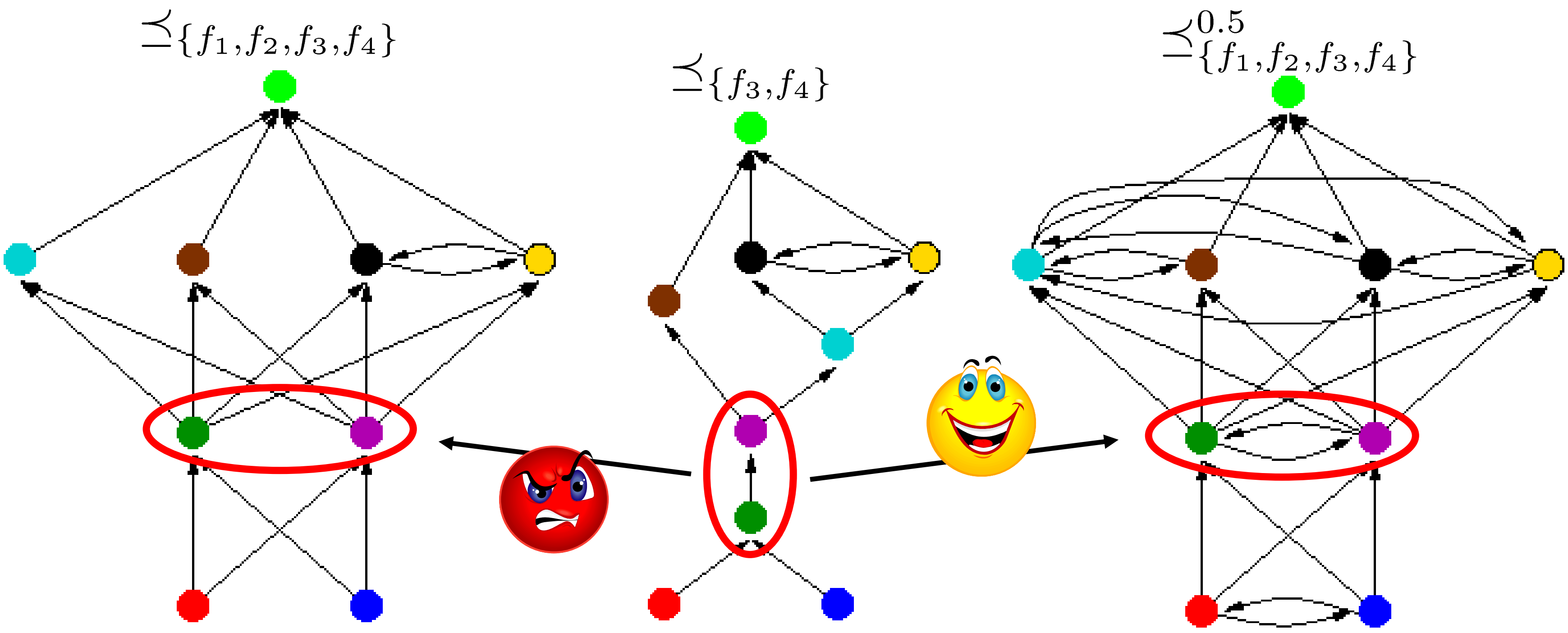
$$\preceq_{\mathcal{F}}^{\delta} := \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x}, \mathbf{y} \in X \wedge \forall f_i \in \mathcal{F} : f_i(\mathbf{x}) - \delta \leq f_i(\mathbf{y})\}$$





δ–Conflict

- **Definition:** $\mathcal{F}_1$ δ-nonconflicting with $\mathcal{F}_2$ iff $\preceq_{\mathcal{F}_1} \subseteq \preceq^{\delta}_{\mathcal{F}_2}$ and $\preceq_{\mathcal{F}_2} \subseteq \preceq^{\delta}_{\mathcal{F}_1}$
- Omission of objectives in $\mathcal{F} \setminus \mathcal{F}'$ if $\mathcal{F}' \subseteq \mathcal{F}$ is δ-nonconflicting with $\mathcal{F}$ guarantees that $\mathbf{x} \preceq^{\delta}_{\mathcal{F}} \mathbf{y}$ whenever $\mathbf{x} \preceq_{\mathcal{F}'} \mathbf{y}$



Key Contributions

- Generalization of Objective Conflicts
- **The Minimum Objective Subset Problems**
  - **Exact and heuristic algorithms**
- Objective reduction for selected problems

The Minimum Objective Subset Problem

Minimum objective set

$\mathcal{F}' \subseteq \mathcal{F}$ is called minimum if $\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}$

and $\nexists \mathcal{F}'' \subseteq \mathcal{F} \wedge |\mathcal{F}''| < |\mathcal{F}'| : \preceq_{\mathcal{F}''} = \preceq_{\mathcal{F}}$

Minimum Objective Subset Problem (MOSS)

Given: Set A of solutions with weak dominance relations $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$ and $\preceq_{f_i} \subseteq A \times A$

Task: Compute a minimum objective set $\mathcal{F}' \subseteq \mathcal{F}$ with $\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}$

MOSS is NP-hard

- Reduction from set cover problem (SCP)
- As a result, consideration of objective sets of fixed size is not sufficient

δ–Minimum objective set

$\mathcal{F}' \subseteq \mathcal{F}$ is called δ-minimum if $\preceq_{\mathcal{F}'} = \preceq^{\delta}_{\mathcal{F}}$, $\forall \delta' < \delta : \preceq_{\mathcal{F}'} \neq \preceq^{\delta'}_{\mathcal{F}}$
and $\nexists \mathcal{F}'' \subseteq \mathcal{F} \wedge |\mathcal{F}''| < |\mathcal{F}'| : \preceq_{\mathcal{F}''} = \preceq^{\delta}_{\mathcal{F}}$

δ–Minimum Objective Subset Problem (δ-MOSS)

Given: Set A of solutions with weak dominance relations $\preceq_{f_i} \subseteq A \times A$ and $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$ and a $\delta \geq 0$

Task: Compute a δ-minimum objective set $\mathcal{F}' \subseteq \mathcal{F}$ wrt $\mathcal{F}$

Objective Subset of size k with minimum error (kEMOSS)

Given: Set A of solutions with weak dominance relations $\preceq_{f_i} \subseteq A \times A$ and $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$ and a k

Task: Compute an objective subset $\mathcal{F}' \subseteq \mathcal{F}$, δ–nonconflicting with $\mathcal{F}$, $|\mathcal{F}'| \leq k$ and minimal δ

Exact algorithm

- Correctness proof
- Runtime: $O(|A|^2 \cdot k \cdot 2^k)$
- Worst case: $\Omega(|A|^2 \cdot 2^{k/3})$

Init:
  $M := \emptyset, \quad S_M := \emptyset$
for all pairs $\mathbf{x}, \mathbf{y} \in A$, $\mathbf{x} \neq \mathbf{y}$ of solutions **do**
  $S_{\{(\mathbf{x},\mathbf{y})\}} := \emptyset$
  for all objective pairs $i, j \in \mathcal{F}$, not necessary $i \neq j$ **do**
    compute $\delta_{ij} := \delta_{\min}(\{i\} \cup \{j\}, \mathcal{F})$ wrt $\mathbf{x}, \mathbf{y}$
    $S_{\{(\mathbf{x},\mathbf{y})\}} := S_{\{(\mathbf{x},\mathbf{y})\}} \sqcup (\{i\} \cup \{j\}, \delta_{ij})$
  end for
  $S_{M \cup \{(\mathbf{x},\mathbf{y})\}} := S_M \sqcup S_{\{(\mathbf{x},\mathbf{y})\}}$
  $M := M \cup \{(\mathbf{x}, \mathbf{y})\}$
end for
Output for δ-MOSS:
  $(s_{\min}, \delta_{\min})$ in S_M with minimal size $|s_{\min}|$ and $\delta_{min} \leq \delta$
Output for **k**EMOSS:
  (s, δ) in S_M with size $|s| \leq \mathbf{k}$ and minimal δ

Simple greedy heuristics

- Correctness proof
- Runtime
  - $O(\min\{k^3 \cdot |A|^2, k^2 \cdot |A|^4\})$ (δ MOSS)
  - $O(k^3 \cdot |A|^2)$ (kEMOSS)
- Best possible approximation ratio of $\Theta(\log |A|)$ for the case $\delta = 0$

δ -MOSS

Init:
  compute the relations $\preceq_i$ for all $1 \leq i \leq k$ and $\preceq_{\mathcal{F}}$
  $\mathcal{F}' := \emptyset$
  $R := A \times A \setminus \preceq_{\mathcal{F}}$
while $R \neq \emptyset$ **do**
  $i^* = \operatorname{argmin}_{i \in \mathcal{F} \setminus \mathcal{F}'} \{|(R \cap \preceq_i) \setminus (\preceq^0_{\mathcal{F}' \cup \{i\}} \cap \preceq^\delta_{\mathcal{F} \setminus (\mathcal{F}' \cup \{i\})})|\}$
  $R := (R \cap \preceq_{i^*}) \setminus (\preceq^0_{\mathcal{F}' \cup \{i^*\}} \cap \preceq^\delta_{\mathcal{F} \setminus (\mathcal{F}' \cup \{i^*\})})$
  $\mathcal{F}' := \mathcal{F}' \cup \{i^*\}$
end while

kEMOSS

Init:
  $\mathcal{F}' := \emptyset$
while $|\mathcal{F}'| < \mathbf{k}$ **do**
  $\mathcal{F}' := \mathcal{F}' \cup \operatorname{argmin}_{i \in \mathcal{F} \setminus \mathcal{F}'} \{\delta_{\min}(\mathcal{F}' \cup \{i\}, \mathcal{F})$ wrt $A\}$
end while

Key Contributions

- Generalization of Objective Conflicts
- The Minimum Objective Subset Problems
  - Exact and heuristic algorithms
- **Objective reduction for selected problems**

Objective Reduction for Selected Problems

No error:

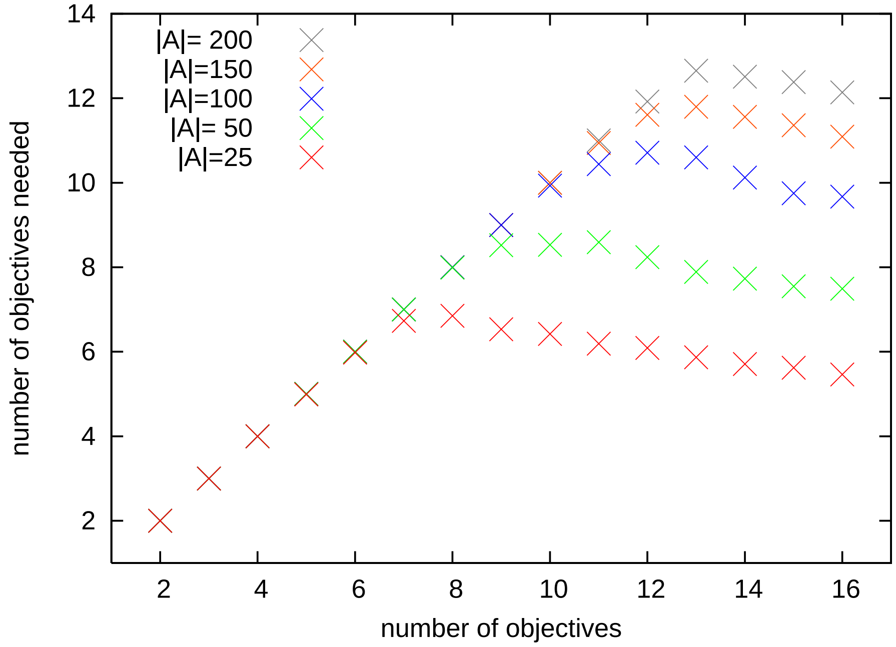
- Solutions with randomly chosen objective values (i.e., random orders as $\preceq_{f_i}$):
  - Objective reduction possible?
  - Size of minimum set influenced by solution set size and number of objective?
  - Greedy vs. exact algorithm
- Realistic scenarios for test problems

Influence of δ and k:

- Comparison between greedy and exact algorithms

Varying |A| and k for Random Orders

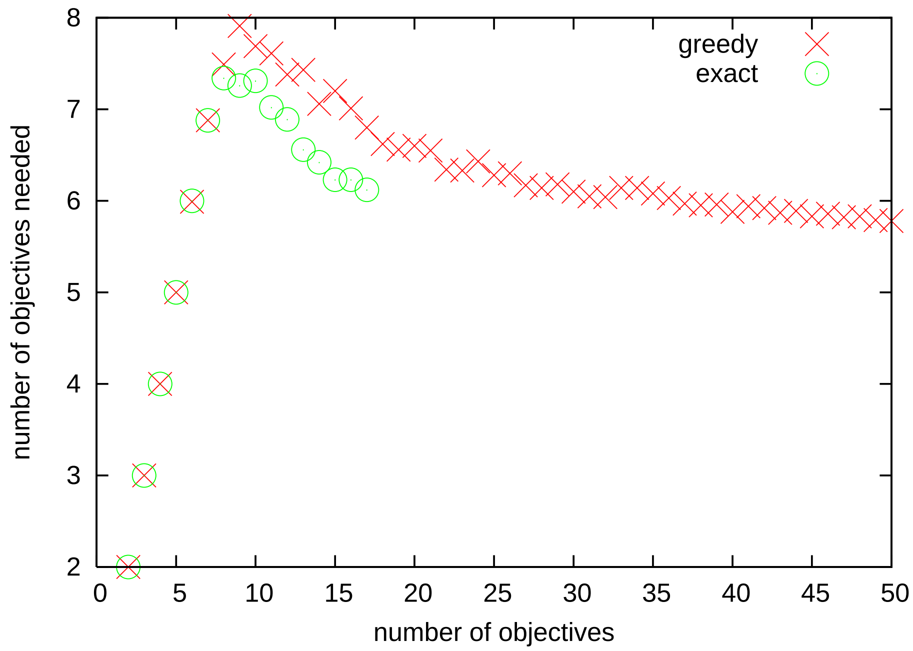
Various solution set sizes $|A|$ with random orders as $\preceq_{f_i}$



- The more objectives, the smaller the minimum sets
- The more solutions in A, the fewer objectives omissable

Greedy vs. Exact Algorithm for Random Orders

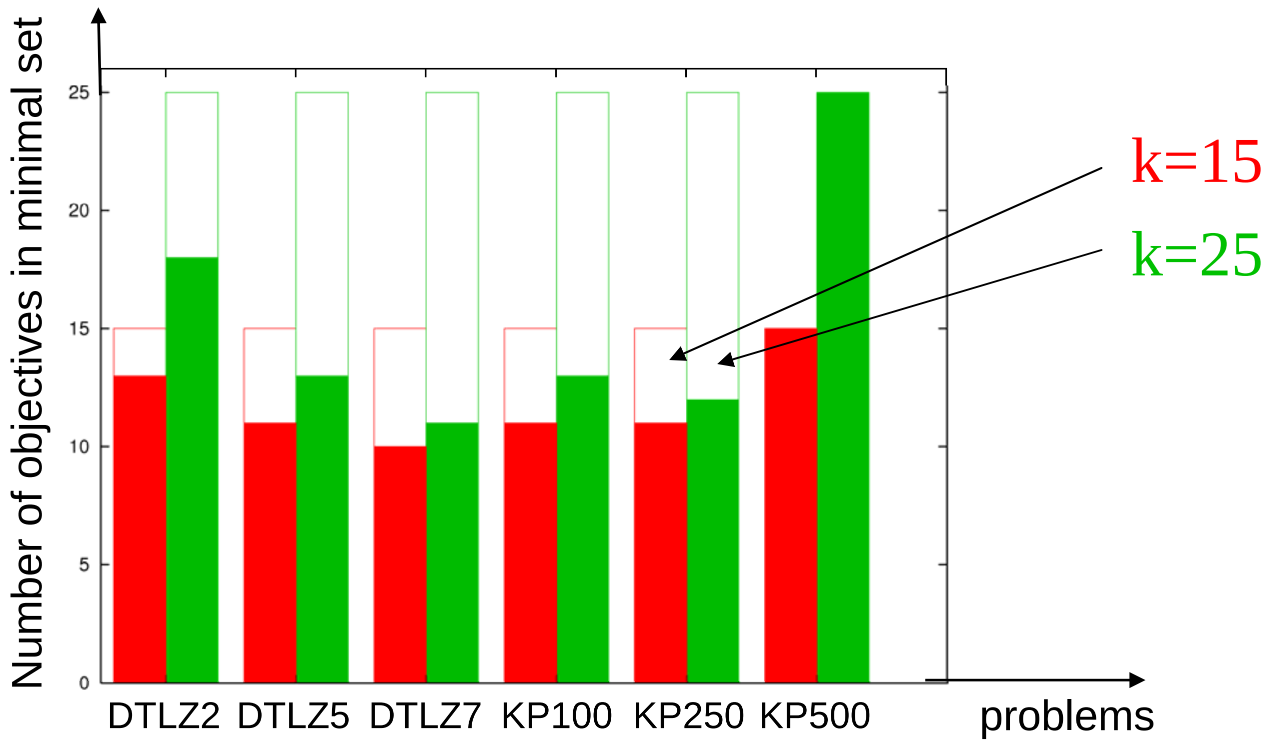
Heuristic vs. exact algorithm on random orders $\preceq_{f_i}$ with $|A| = 32$



- The greedy algorithm's objective sets are not too large
- Greedy algorithm has clearly lower running time:
  - can handle 50 objectives instead of ≤ 20 compared to exact algorithm within the same time

Realistic Scenarios for Test Problems

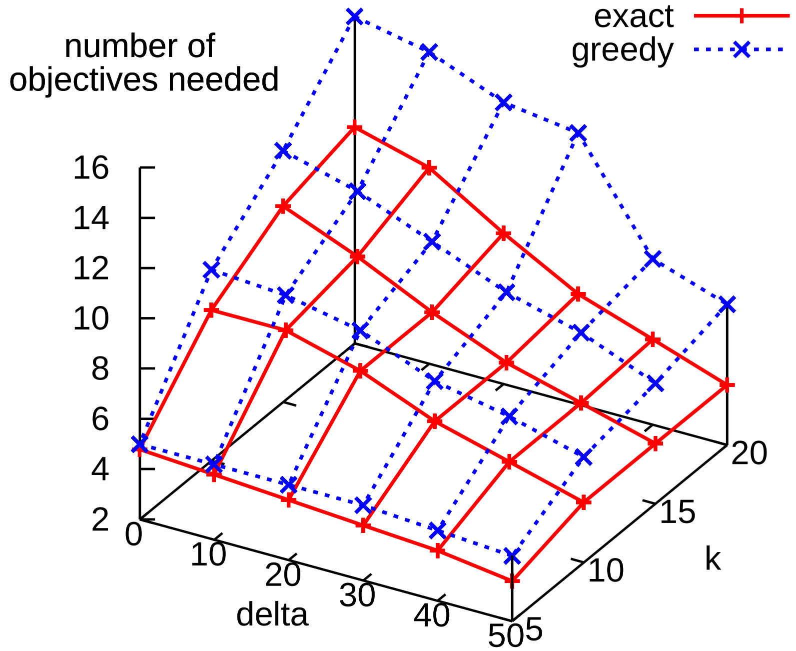
- Approximation of efficient set computed by evolutionary algorithm used as A
- $|A| = 200$ for $k = 15$ and $|A| = 300$ for $k = 25$

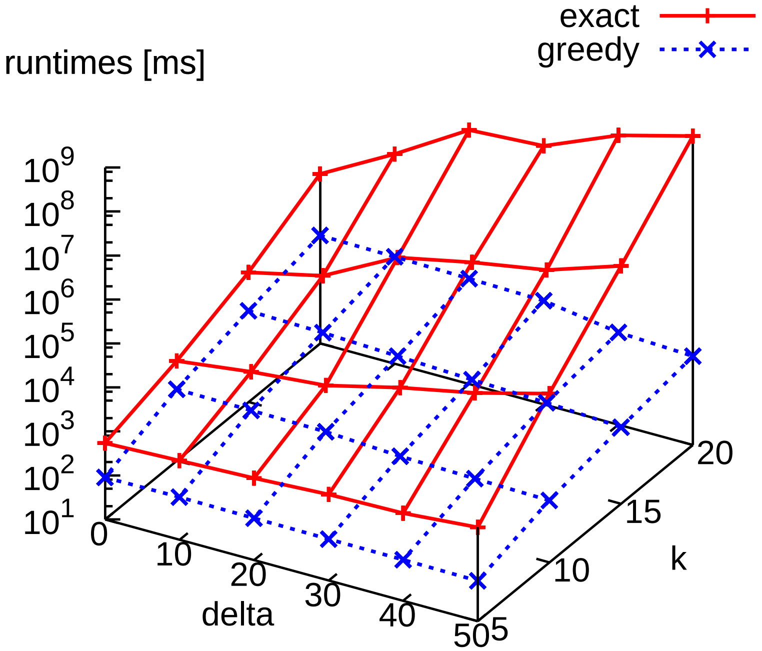


- Objective reduction of $\leq$ 50% possible for various test problems

Comparison of Algorithms for δ-MOSS

Entire Search Space of 0-1-Knapsack Problem with 7 Items





- heuristic slightly worse results, but clearly faster

⇒
- the more objectives, the more objectives can be omitted
- The larger the error, the smaller the sets

Outline

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Problems with Many Objectives

- MOEAs, working on 2D and 3D problems are not suitable for many objective optimization (NSGA-II, SPEA2, …)
  - Why?
    - Not clear in general
    - Number of incomparable solution pairs increases

- Widely believed, that problems become harder with more objectives

Related Work

Reducing number of objectives:

- Maneeratana et al. (2006):
  - Reducing of MOP to 2D-problem (drawback: new objectives, no preservation of dominance relation)
- Deb and Saxena (2005):
  - Multiple starts of NSGA-II with reduced number of objectives, choice of objectives based on PCA

General Investigations:

- Neumann and Wegener (2006), Scharnow et al. (2002):
  - Few examples where more objectives help
- But nearly every textbook says that more objectives makes the problem harder, e.g., Deb (2001)
- P. Winkler (1985):
  - Random orders as objectives with n points in k dimensions
  - Width between $e^{-1}n^{(k-1)/k}$ and $n^{(k-1)/k}\ln(n)$

Key Questions

Reducing Number of Objectives Within Search:

- How to include (adaptive) objective reduction into EA while using subset of given objectives?

General Investigations:

- Does all problems become harder with more objectives?
- Is it due to more incomparable solutions?

Reducing Number of Objectives Within Search

- If EA detects, that objectives can be omitted, then objective reduction is not necessary any more
  - Exception: objective function evaluations are expensive
- Problem is not the number of objectives but the number of incomparable solutions
  - No direction to better solutions observable
  - Potential way out:
    - use indicator to refine Pareto dominance relation (e.g. Hypervolume indicator/S-metric/Lebesgue-measure)

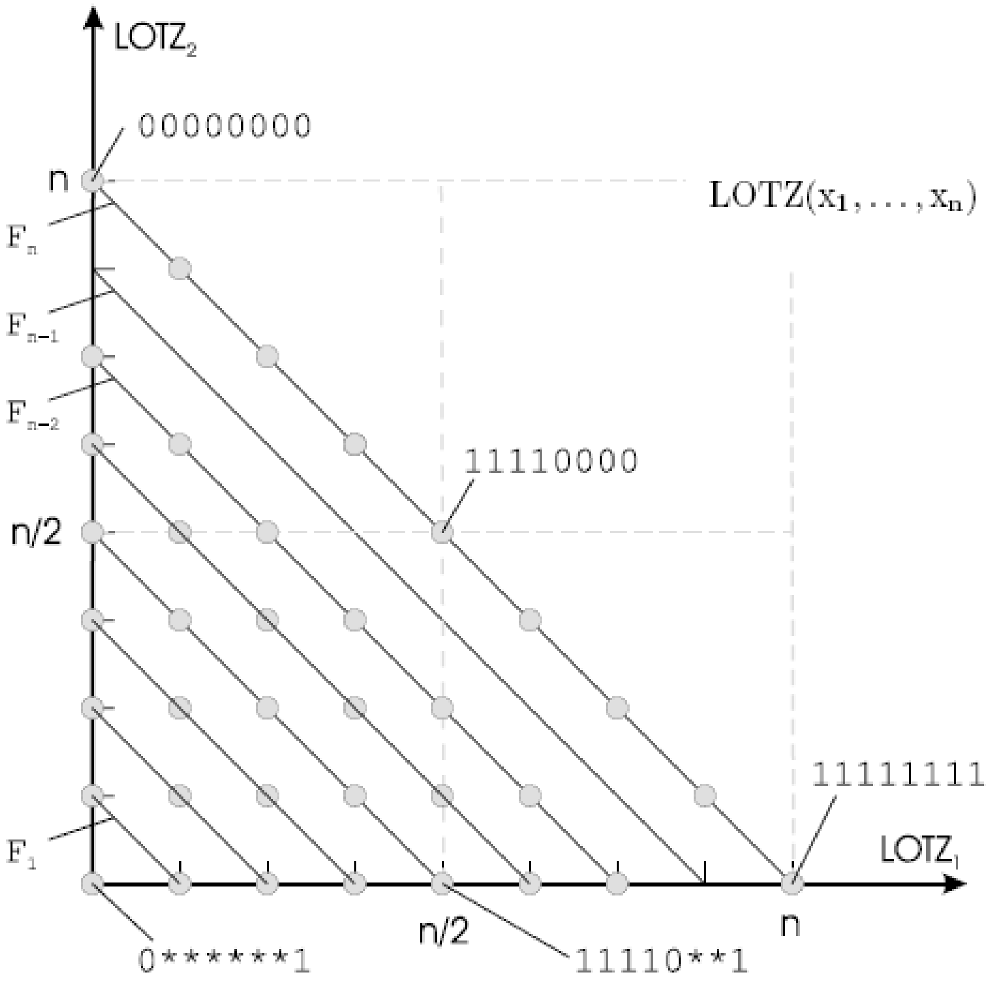
General Investigations

Do all problems become harder with more objectives?

Is it due to more incomparable solutions?

- 4 simple (toy) problems based on 2D problem
- LOTZ, resp. modified LOTZ
- Add third objective
  - This can both increase or decrease the difficulty of the problem
  - Both when
    - making indifferent solutions comparable, and
    - making comparable solutions incomparable!

LOTZ - Leading Ones Trailing Zeros



$$\mathrm{LOTZ}(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} \prod_{j=1}^{i} x_j, \sum_{i=1}^{n} \prod_{j=i}^{n} (1 - x_j) \right)$$

Third Objectives Makes Indifferent Comparable

- Problem 1 (harder than LOTZ):

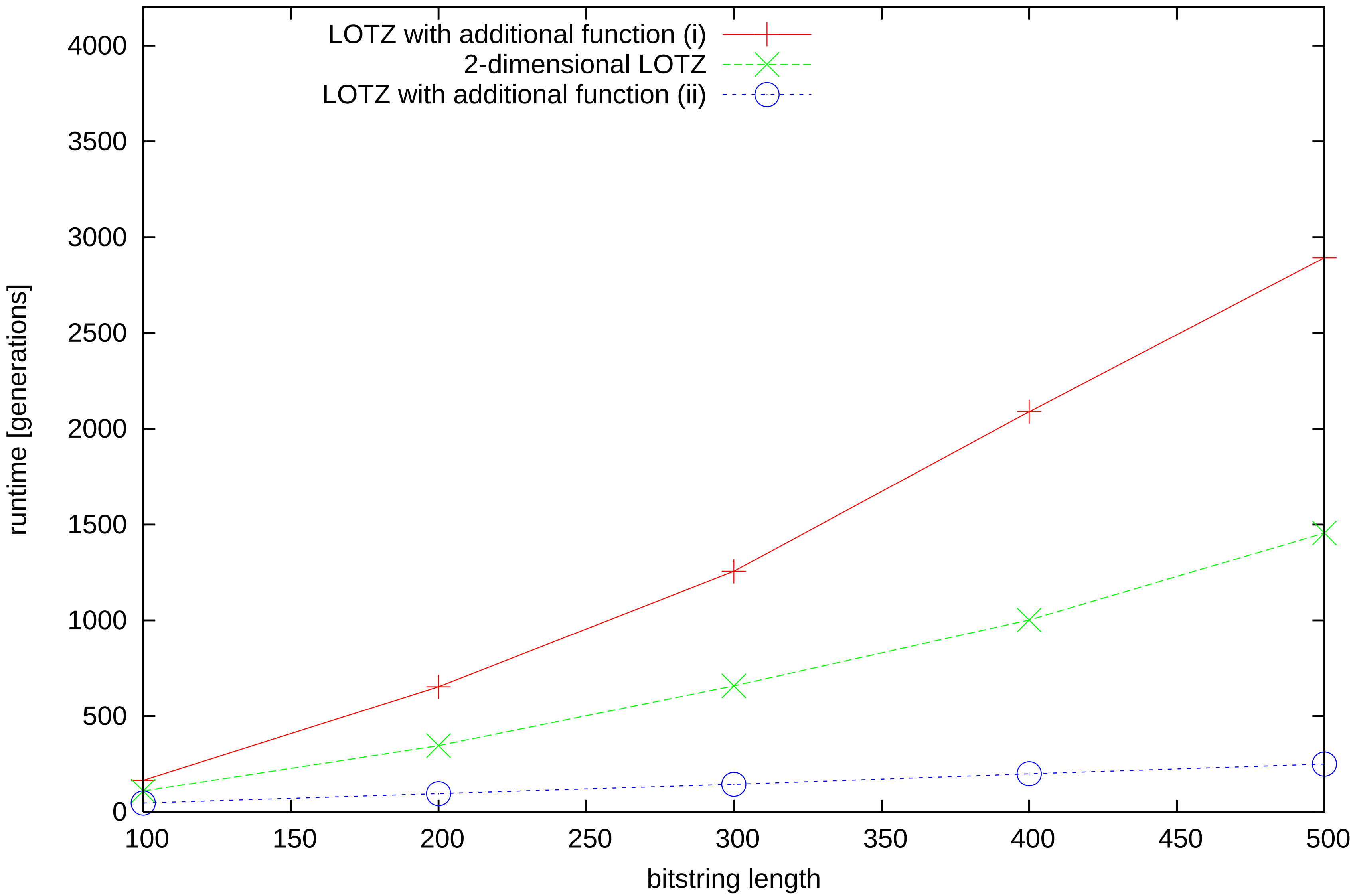
$$\begin{aligned} f_1(\mathbf{x}) &:= \texttt{LEADING ONES}(\mathbf{x}) \\ f_2(\mathbf{x}) &:= \texttt{TRAILING ZEROS}(\mathbf{x}) \\ f_3(\mathbf{x}) &:= |\mathbf{x}_M| - \texttt{LEADING ONES}(\mathbf{x}_M) - \texttt{TRAILING ZEROS}(\mathbf{x}_M) \end{aligned}$$

- Problem 2 (easier than LOTZ):

$$\begin{aligned} f_1(\mathbf{x}) &:= \texttt{LEADING ONES}(\mathbf{x}) \\ f_2(\mathbf{x}) &:= \texttt{TRAILING ZEROS}(\mathbf{x}) \\ f_3(\mathbf{x}) &:= \texttt{ONEMAX}(\mathbf{x}_M) \end{aligned}$$

Third Objectives Makes Indifferent Comparable (2)

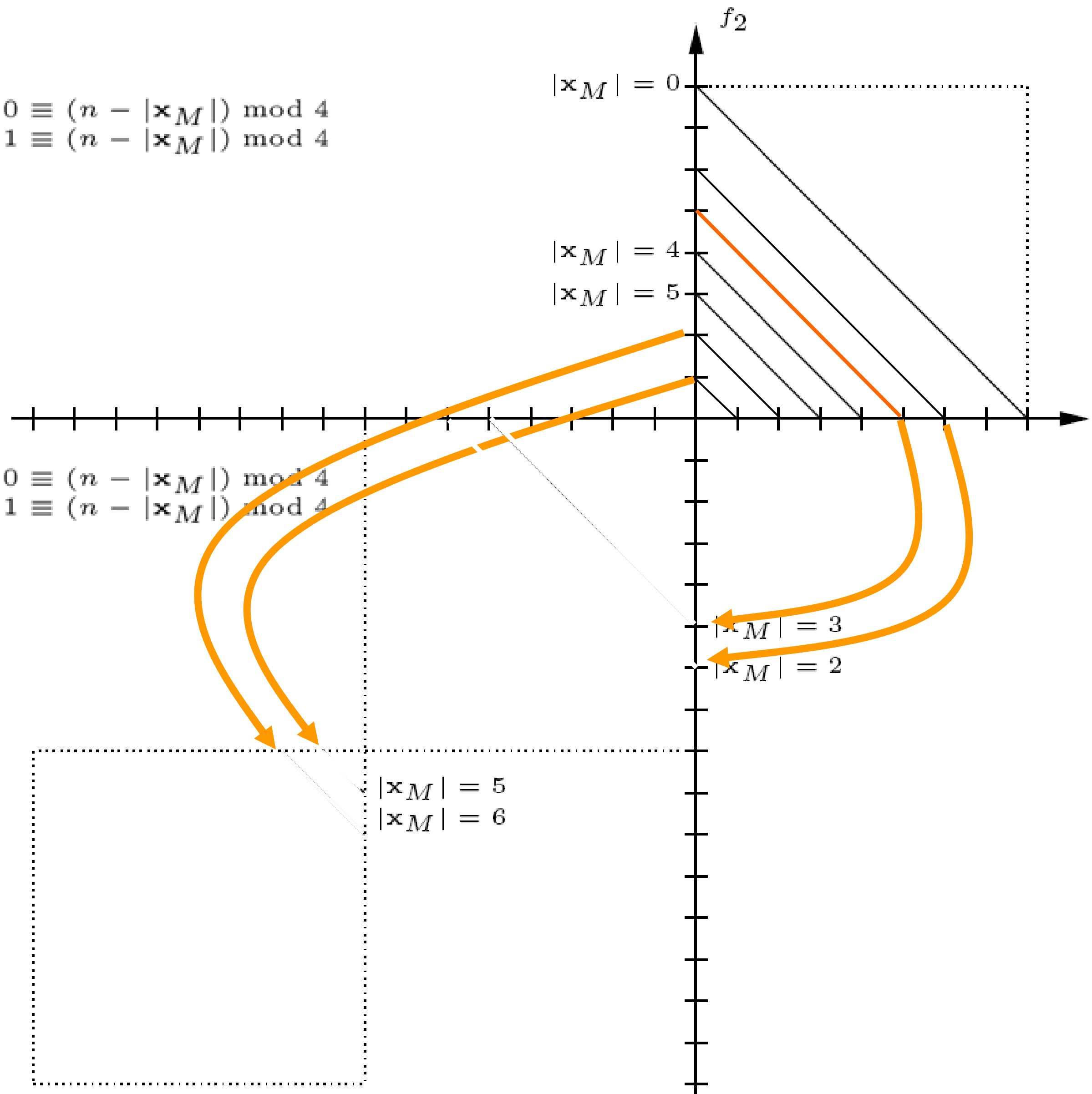
Average runtimes for 10 IBEA runs with population size 200



Modified LOTZ

$$\texttt{moLO}(\mathbf{x}) = \begin{cases} \texttt{LO}(\mathbf{x}) & \text{iff} \quad 0 \equiv (n - |\mathbf{x}_M|) \bmod 4 \\ & \text{or} \quad 1 \equiv (n - |\mathbf{x}_M|) \bmod 4 \\ (-n) \cdot \dfrac{\frac{n-|\mathbf{x}_M|}{2}}{2} - \texttt{LO}(\mathbf{x}) & \text{else} \end{cases}$$

$$\texttt{moTZ}(\mathbf{x}) = \begin{cases} \texttt{TZ}(\mathbf{x}) & \text{iff} \quad 0 \equiv (n - |\mathbf{x}_M|) \bmod 4 \\ & \text{or} \quad 1 \equiv (n - |\mathbf{x}_M|) \bmod 4 \\ (-n) \cdot \dfrac{\frac{n-|\mathbf{x}_M|}{2}}{2} - \texttt{TZ}(\mathbf{x}) & \text{else} \end{cases}$$



Third Objectives Makes Comparable Incomparable

- Problem 3 (harder than modified LOTZ):

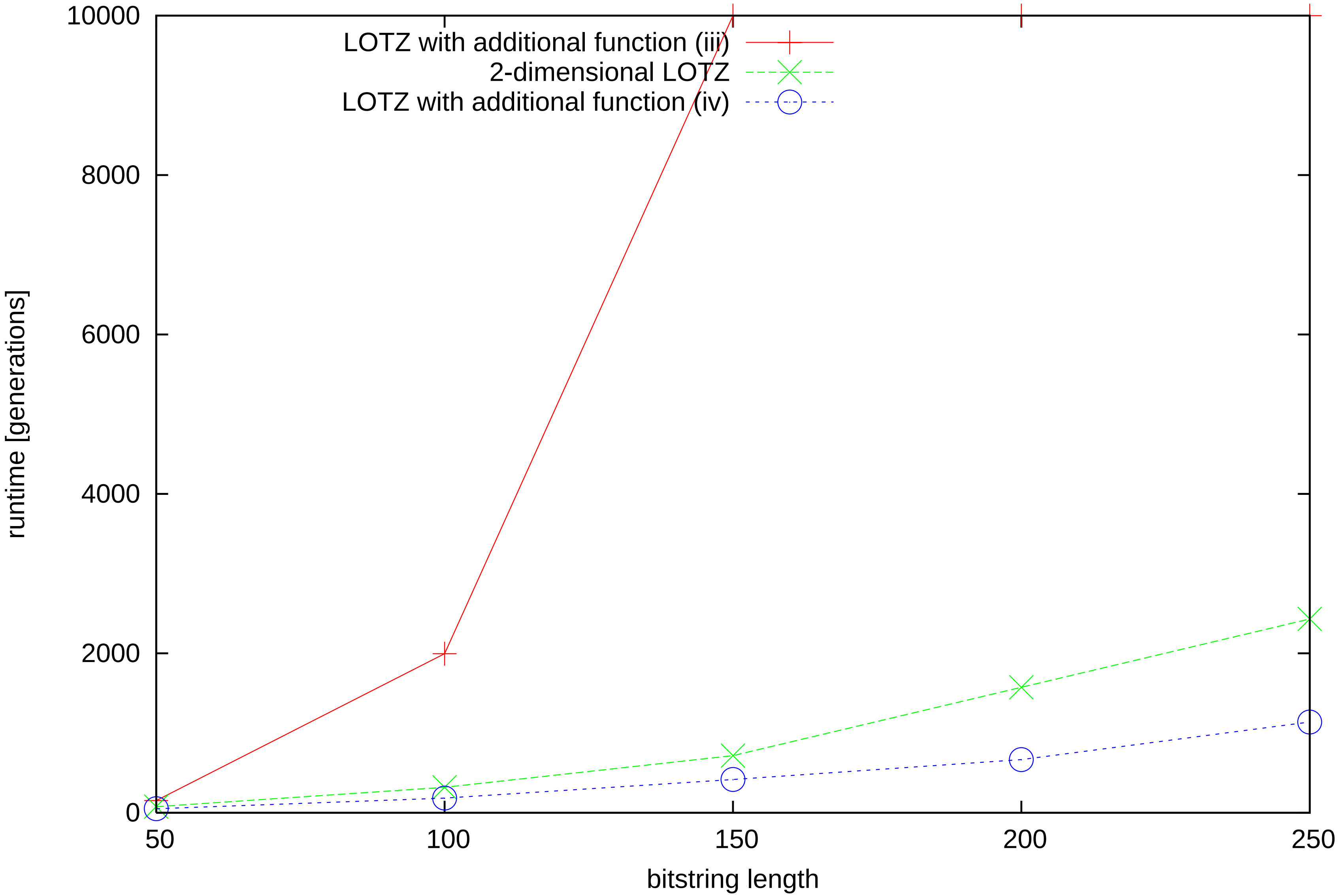
$$\begin{aligned} f_1(\mathbf{x}) &:= \text{modified } \texttt{LEADING ONES}(\mathbf{x}) \\ f_2(\mathbf{x}) &:= \text{modified } \texttt{TRAILING ZEROS}(\mathbf{x}) \\ f_3(\mathbf{x}) &:= \frac{n}{2} - |\frac{n}{2} - |\mathbf{x}_M|| \end{aligned}$$

- Problem 4 (easier than modified LOTZ):

$$\begin{aligned} f_1(\mathbf{x}) &:= \text{modified } \texttt{LEADING ONES}(\mathbf{x}) \\ f_2(\mathbf{x}) &:= \text{modified } \texttt{TRAILING ZEROS}(\mathbf{x}) \\ f_3(\mathbf{x}) &:= |\mathbf{x}_M| \end{aligned}$$

Third Objectives Makes Comparable Incomparable (2)

Average runtimes for 10 IBEA runs with population size 100



Conclusions and Outlook

- Generalization of Objective Conflicts
- The MOSS Problem and algorithms
- Method feasable for decision making process for selected problems
  - Also for real world problems?
- General discussion of problems with many objectives
  - Current work: general indicator properties

Literature

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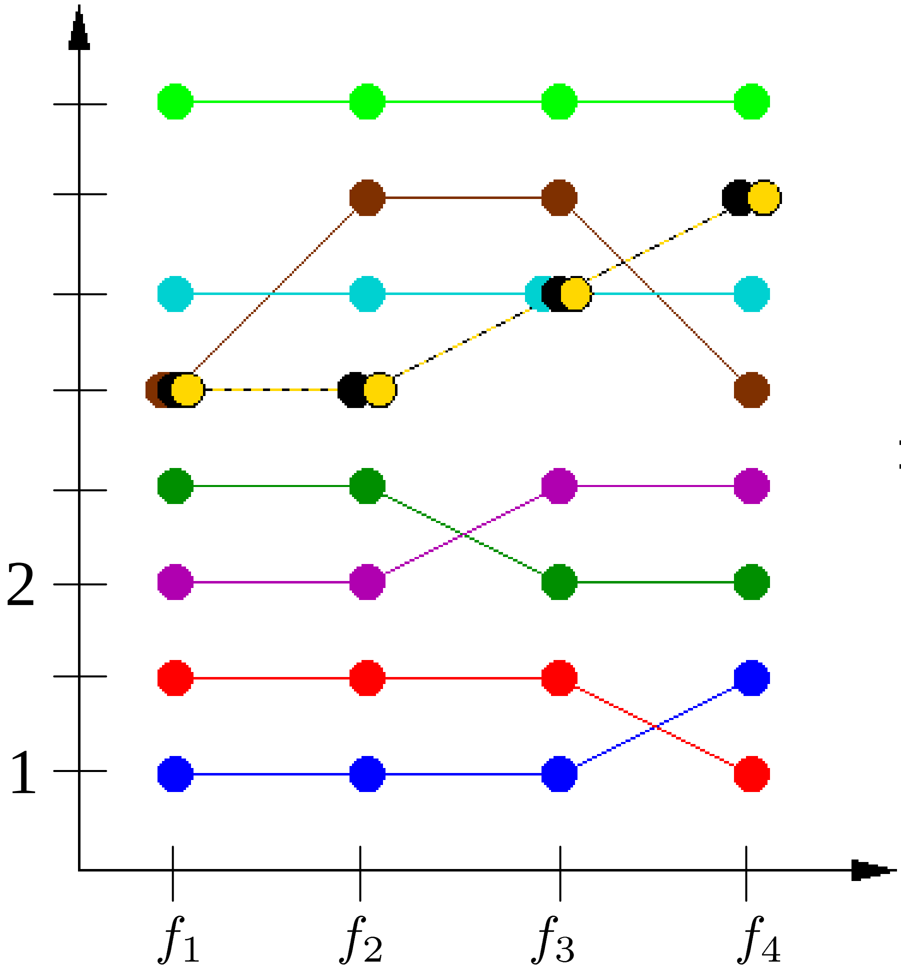
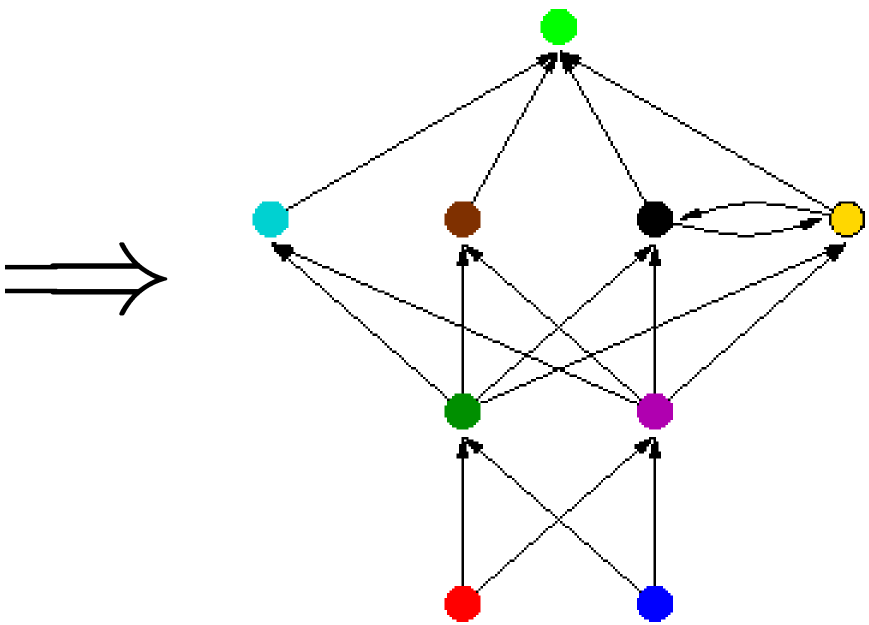
Parallel Coordinates Plot for Example





Parallel Coordinates Plot for Example