Dynamic Min-Max Problems

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Abstract

In this paper we address min-max equations for periodic and non-periodic problems. In the non-periodic case a simple algorithm is presented to determine whether a graph has a potential satisfying the min-max equations. This method can also be used to solve a more general min-max problem on periodic graphs. Also some results regarding the uniqueness of solutions in the periodic case are given. Finally, we address a more general quasi periodic problem and provide an algorithm for its solution.

Key words. Min-max equations, graph theory, periodic graphs

1 Introduction

Problems most closely related to the results presented in our paper are network flow problems (non-periodic case), see e.g. [1], and the well known maximum cycle mean problem (periodic case), see e.g. [14], [12]. In particular, known results in the *non-periodic case* can be related to a feasible potential function p observing lower linear constraints

$$p(v_j) \leq p(v_i) + w(v_i, v_j) \quad \forall v_j \in V^-, (v_i, v_j) \in E \tag{1}$$

and an optimal potential function p using min constraints

$$p(v_j) = \min\{p(v_i) + w(v_i, v_j) \mid (v_i, v_j) \in E\} \quad \forall v_j \in V^- \tag{2}$$

associated with some network $\mathcal{G}^-(V^-, E, w)$. Our paper addresses a generalization where these sets of inequalities are mixed for a given network $\mathcal{G}(V, E, w)$ with their dual forms, that is upper linear constraints

$$p(v_j) \geq p(v_i) + w(v_i, v_j) \quad \forall v_j \in V^+, (v_i, v_j) \in E \tag{3}$$

and max constraints

$$p(v_j) = \max\{p(v_i) + w(v_i, v_j) \mid (v_i, v_j) \in E\} \quad \forall v_j \in V^+ \tag{4}$$

with $V^+ \cup V^- = V$. If a distance function d is given additionally, the corresponding *periodic problems* deal with edge weights $w(v_i, v_j) - \lambda d(v_i, v_j)$ using the period λ as a parameter. Moreover, we define a quasi-periodic problem with edge weights $w(v_i, v_j) - \lambda(v_j) d(v_i, v_j)$ where $\lambda(v_j)$ is the specific period associated with node v_j.

1.1 Results and Applications in the Non-periodic Case

In the area of interface timing verification, see [15], [21], [22], frequently problems related to the existence of min and/or max constraints occur. There, the difference between the potentials of two nodes must be maximized under various constraints. In particular, it is possible to transform one of the problems addressed in [15], [21] and [22] to a problem with mixed constraints (1), (2) and (3). Different pseudo-polynomial algorithms are derived for the solution of this problem based on iterative tightening [15], removing negative cycles [21] and maximum separations [22]. However so far, neither a polynomial algorithm nor a proof of intractability is known.

In comparison to these results, we are mainly dealing with constraints (2) and (4). Note that constraints of the form (1) or (3) can easily be converted into constraints (2) and (4) by a simple transformation of the graph. In particular, one additional node and two edges must be added for each node v_j with constraints of type (1) or (3). With respect to the non-periodic case our paper contains the following results:

- A relation between potential functions based on (2,4) and those on (3, 2) and (1, 4) is given.
- We present efficient pseudo-polynomial algorithms for finding optimal potentials satisfying constraints (2) and (4).

1.2 Results and Applications in the Periodic Case

The consideration of constraints (4) in connection with periodic graphs has raised significant interest in the past as it is the root for many problems from different application areas. This includes e.g. control theory and manufacturing [7], timing properties of discrete event systems [4], [19]), parallel algorithms [20], [9], and other areas of computer science. A comprehensive treatment of the theory and applications can be found in [2]. Many results have been developed while considering linear equations over a new max-plus algebra, see [7], [8], [23] and [2]. One of the main results is the establishment of a relation between the asymptotic period of a dynamic graph, the eigenvalue of the weight matrix of the corresponding static graph $\mathcal{G}$ and the maximum cycle mean of the static graph. Different algorithms are known to compute this value, notably an algorithm based on binary search [14] and a polynomial algorithm discovered by Karp [12]. For dynamic graphs several other combinatorial problems have been investigated as well, see e.g. [5], [13], [18]. Some of these results have even been generalized to problems which are periodic in multiple dimensions, see [3].

Driven by application areas like asynchronous circuit design, timing and protocol verification and timing behavior of general Petri nets there have been recently several attempts to generalize these results to dynamic graphs with constraints of the form (4) *and* (2). These dynamic min-max systems have been investigated in [16], [17] and [2]. Further results in this direction are described in [10] and [11]. However, the models used in these investigations are quite different. Olsder [16, 17] describes a periodic min-max problem in terms of an eigenvalue problem, whereas Gunawardena [10, 11] defines a certain class of min-max functions. Both models are special cases of those used in our paper. In particular, the model proposed by Olsder corresponds to the case that $d(v_i, v_j) = 1$ while that of Gunawardena can be obtained with $d(v_i, v_j) \in \{0, 1\}$.

Also, results in [16], [17] with respect to the uniqueness of the period and the numerical procedures are restricted to a subclass of min-max problems. On the other hand, [10] and [11] contain "complete" results in the case that only two distances have the value 1 while all others are zero. For all other considered cases ($d(v_i, v_j) \in \{0, 1\}$), there is no procedure which decides whether a min-max system has a period or not. Moreover, the given algorithm for the computation of the period is exponential in the size of the graph.

Our paper contains the following new results:

- A relation between potential functions of dynamic and weight transformed static graphs is derived. This is similar to a known result for max-plus problems [6].
- The first efficient (pseudo-polynomial) algorithm for the determination of the period of min-max systems is presented.
- Results on the existence and uniqueness of the period of a dynamic min-max system are given.
- Results on the uniqueness of the periods in the quasi-periodic case are given as well as algorithms to determine these periods.

2 The Static Min-Max Problem

2.1 Definitions and Properties

We start this section by defining various forms of graph potentials.

Definition 1 (Min-Max Potential) *Assume a weighted digraph $\mathcal{G}(V = V^+ \cup V^-, E, w)$ with $V^+ \cap V^- = \emptyset$, $E \subseteq V \times V$ and $w : E \to \mathcal{Q}$, also called min-max graph subsequently. Then, a potential $p : V \to \mathcal{Q}$ is called feasible if*

$$p(v_i) \begin{cases} \geq & p(v_j) + w(v_j, v_i) \quad \forall (v_j, v_i) \in E, v_i \in V^+ \\ \leq & p(v_j) + w(v_j, v_i) \quad \forall (v_j, v_i) \in E, v_i \in V^- \end{cases}$$

Further, a feasible potential $p : V \to \mathcal{Q}$ is a min potential if

$$p(v_i) = \min(p(v_j) + w(v_j, v_i) \mid (v_j, v_i) \in E) \qquad \forall v_i \in V^-$$

Similarly, a feasible potential $p : V \to \mathcal{Q}$ is a max potential if

$$p(v_i) = \max(p(v_j) + w(v_j, v_i) \mid (v_j, v_i) \in E) \qquad \forall v_i \in V^+$$

Finally, a potential $p : V \to \mathcal{Q}$ is a min-max potential if it is a min potential and a max potential at the same time.

The definition of a min-max potential directly leads to our first key problem:

Problem 1 *Is there a min-max potential for a given min-max graph $\mathcal{G}$?*

Before addressing this problem a few statements can provide some help to simplify the problem:

1. If $\mathcal{G}$ consists of two independent graphs it is sufficient to consider each graph separately.
2. If $\mathcal{G}^+ = (V, E \cap (V^+ \times V^+), w)$ contains a positive weight cycle then there is no min-max potential for $\mathcal{G}$.
3. If $\mathcal{G}^- = (V, E \cap (V^- \times V^-), w)$ contains a negative weight cycle then there is no min-max potential for $\mathcal{G}$.

Further, it suffices to consider only bipartite min-max graphs where $E \subseteq (V^+ \times V^-) \cup (V^- \times V^+)$ as shown in the next corollary.

Corollary 1 *Assume a min-max graph $\mathcal{G}(V, E, w)$ without non-negative weight cycles among nodes from V^+ and non-positive weight cycles among nodes from V^-. Then there is a bipartite min-max graph $\mathcal{G}_b(V, E_b \subseteq (V^+ \times V^-) \cup (V^- \times V^+), w_b)$ such that there is a min-max potential for G_b if and only if there is a min-max potential for $\mathcal{G}$.*

Proof: Consider all simple paths $(v_s = v_0, v_1, \ldots, v_{k-1}, v_t = v_k)$ between a node $v_s \in V^-$ and a node $v_t \in V^+$ in $\mathcal{G}$ such that $v_i \in V^+$ for all $1 \leq i \leq k$. If there is at least one such path then introduce an edge (v_s, v_t) in E_b with edge weight $w_b(v_s, v_t) = \max_{\text{all such paths}} \sum_{i=0}^{k-1} w(v_i, v_{i+1})$. If there is a max potential for $\mathcal{G}$ there must be one of the above described paths for any $v_t \in V^+$ with $p(v_{i+1}) = p(v_i) + w(v_i, v_{i+1})$ for all $0 \leq i < k$ as there are no zero weight cycles among nodes from V^+. Therefore, a max potential for $\mathcal{G}$ is also a max potential for G_b. The other direction can be proven in a similar way. Also the same proof can be used for min potentials as well. ■

Note that the existence of zero weight cycles among nodes from V^+ or V^- can be easily checked by the use of shortest or longest path algorithms. Then, by introducing an additional node from V^- or V^+, respectively, into these cycles the conditions for Corollary 1 can be satisfied. Moreover, in-degree$(v) > 0$ for all $v \in V$ is a necessary condition for the existence of a min-max potential. Therefore, we assume for the remainder of this section that $\mathcal{G}$ is a connected bipartite graph and that for each node $v_j \in V$ there is at least one edge $(v_i, v_j) \in E$.

In the next corollary we show that a graph with a min potential will retain this property even if the weights of some edges are reduced.

Corollary 2 *If a bipartite graph $\mathcal{G}(V, E, w)$ has a min potential then any graph $\mathcal{G}'(V, E, w')$ with $w'(v_i, v_j) \leq w(v_i, v_j)$ for all $(v_i, v_j) \in E$ has a min potential as well. On the other hand if a bipartite graph $\mathcal{G}(V, E, w)$ has no min potential then any graph $\mathcal{G}'(V, E, w')$ with $w'(v_i, v_j) \geq w(v_i, v_j)$ for all $(v_i, v_j) \in E$ also has no min potential.*

Proof: Let p be a min potential of $\mathcal{G}$ and $w'(v_i, v_j) \leq w(v_i, v_j)$ for all $(v_i, v_j) \in E$. Then p' with

$$p'(v_i) = \begin{cases} p(v_i) & \text{for } v_i \in V^+ \\ \min\{p(v_j) + w'(v_j, v_i) \mid (v_j, v_i) \in E) & \text{for } v_i \in V^- \end{cases}$$

is a min potential for $\mathcal{G}'$ as $p'(v_i) \geq p(v_j) + w(v_j, v_i) \geq p'(v_j) + w'(v_j, v_i)$ for all $v_i \in V^+$ and $(v_j, v_i) \in E$.

The second claim of the corollary is a direct consequence of the first one. ■

Of course, a similar corollary holds for max potentials as well. Now, the confining edges of a min-max potential can be described by use of the tightness graph.

Definition 2 (Tightness Graph) *For any weighted digraph $\mathcal{G}(V, E, w)$ and a potential function $p : V \to \mathcal{Q}$ the tightness graph $\mathcal{G}_p(V, E_p)$ is the graph over V with the edge set $E_p = \{(v_i, v_j) \in E \mid p(v_j) = p(v_i) + w(v_i, v_j)\}$.*

Note that for any min-max potential p, there must be an edge $(v_i, v_j) \in E_p$ for each node $v_j \in V$. It is further easy to see that a min-max potential for a graph $\mathcal{G}$ implies the existence of a cycle C in the corresponding tightness graph. In $\mathcal{G}$ this cycle C must be *zero weight cycle*, i.e. $\sum_{(v_i, v_j) \in C} w(v_i, v_j) = 0$. Some of the definitions given so far are clarified in the examples shown in Figures 1 and 2.

As a min-max potential p for a graph $\mathcal{G}$ is never unique we restrict ourselves to those min-max potentials where $\mathcal{G}_p$ is connected by using Corollary 3.

Corollary 3 *Given a connected bipartite graph $\mathcal{G}$ with min-max potential p. Then there is always a min-max potential p_c such that $\mathcal{G}_{p_c}$ is connected.*

Proof: Assume two unconnected components $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{G}_p$. Then, we use

$$\begin{aligned} E_{12} &= \{(v_i, v_j) \in E \mid v_i \in V^+ \cap \mathcal{G}_1 \text{ and } v_j \in V^- \cap \mathcal{G}_2\} \text{ and} \\ E_{21} &= \{(v_j, v_i) \in E \mid v_i \in V^+ \cap \mathcal{G}_1 \text{ and } v_j \in V^- \cap \mathcal{G}_2\}. \end{aligned}$$

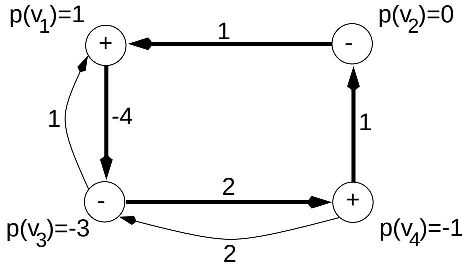

Figure 1: Example of a weighted digraph with a min-max potential. The edge weights $w(v_i, v_j)$ are shown besides the edges. Tight edges, i.e. those in E_p, are bold.

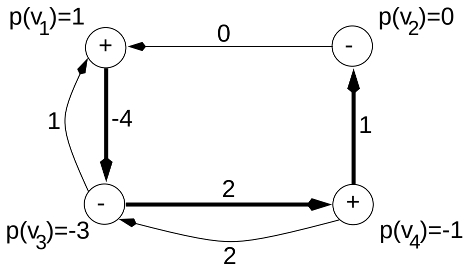

Figure 2: Example of a weighted digraph with a min potential. The edge weights $w(v_i, v_j)$ are shown besides the edges. The graph has no min-max potential. Tight edges, i.e. those in E_p, are bold.

If $E_{12} \cup E_{21} = \emptyset$ then the components $\mathcal{G}_1$ and $\mathcal{G}_2$ are simply exchanged. Next, we define

$$
\begin{aligned}
d_{12} &= \min\{\infty, p(v_j) - p(v_i) - w(v_i, v_j) \mid (v_i, v_j) \in E_{12}\} \\
d_{21} &= \min\{\infty, p(v_j) - p(v_i) + w(v_i, v_j) \mid (v_j, v_i) \in E_{21}\}.
\end{aligned}
$$

Finally, if $p(v_i)$ is reduced by $\min(d_{12}, d_{21})$ for all $v_i \in \mathcal{G}_2$ the resulting potential is still a min-max potential and both components are connected.

A generalization to more components is straight forward. ∎

Next, we define an upper bound s for the length of any simple path in $\mathcal{G}$:

$$
s = \sum_{v_j \in V} (\max_{(v_i, v_j) \in E} (|w(v_i, v_j)|)). \tag{5}
$$

Therefore, Corollary 3 states that any min-max potential of $\mathcal{G}$ can be transformed into a min-max potential p with $|p(v_i) - p(v_j)| \leq s$ for all $v_i, v_j \in V$. The same holds for min potentials and max potentials.

Boolean Function simple-increase($\mathcal{G}, p, \mathcal{G}_t$) **{**
    in $\mathcal{G}$; **inout** p; **out** $\mathcal{G}_t$;
    $a = \max(p(v) \mid v \in V^+)$;
loop: $p(v_j) = \min(p(v_i) + w(v_i, v_j) \mid (v_i, v_j) \in E)$ for all $v_j \in V^-$;
    if $(\exists (v_j, v_i) \in E$ with $v_i \in V^+$ and $p(v_i) < p(v_j) + w(v_j, v_i))$ **{**
        $p(v_i) = p(v_j) + w(v_j, v_i)$; **}**
    else { $\mathcal{G}_t = \emptyset$; **return** 'true'; **}**
    if (there is no change in the potential of any node v_i with $p(v_i) \leq a + s$) **{**
        $\mathcal{G}_t$ = subgraph of $\mathcal{G}$ induced by all nodes with $p(v_i) > a + s$;
        return 'false'; **}**
    goto loop;
}

Table 1: Function *simple-increase*

2.2 Algorithms

Now, we describe a method to determine whether a bipartite weighted digraph has a min-max potential. This method is based on Function *simple-increase* in Table 1.

If the initial potential p_{init} is feasible then $p(v)$ does not decrease for any node during the execution of the loop while $\sum_{v \in V^+} p(v)$ increases in every iteration but the last. Otherwise $p(v)$ may only decrease in the first statement of the first iteration for some nodes $v \in V^-$ which can be neglected. Therefore, the function definitely terminates.

Also note that Function *simple-increase* returns a subgraph $\mathcal{G}_t$ of $\mathcal{G}$. This subgraph is of relevance in the dynamic case only and will be discussed later.

Corollary 4 *If and only if the bipartite min-max graph $\mathcal{G}$ has a min potential then Function* simple-increase *returns 'true' and the generated potential p is a min potential.*

Proof: Assume a min potential p' of $\mathcal{G}$ such that $p'(v) \geq p_{init}(v)$ for all nodes $v \in V$. Note that such a min potential always exists if $\mathcal{G}$ has a min potential. Then, we show by induction that $p(v) \leq p'(v)$ for all $v \in V$ during the execution of the loop in Function *simple-increase*.

Obviously, this condition is true at the start of the loop. The validity of this condition for all nodes $v \in V$ implies its validity for all nodes $v \in V^+$ after the first statement in the loop. The same observation can be made for the modification of nodes in V^- afterwards.

On the other hand if Function *simple-increase* returns 'true' then the generated potential is clearly a min potential. ■

Further, any change of the potential of a node $v_i \in V^+$ requires that at some time during the execution of the function there was a node $v_j \in V^-$ with $p(v_i) = p(v_j) + w(v_j, v_i)$. On the other hand if $p(v_i) = p(v_j) + w(v_j, v_i)$ at any time during the execution of the loop for a node $v_i \in V^+$ then there is an edge $(v_k, v_i) \in E_p$ for some node $v_k \in V^-$ provided the function returns 'true'. Hence, if Function *simple-increase* starts with a max potential and returns 'true' the generated potential p will be a min-max potential.

Therefore, we can state the following theorem.

Theorem 1 *$\mathcal{G}$ has a min-max potential if and only if it has a min potential and a max potential.*

Proof: Obviously, a min-max potential of $\mathcal{G}$ implies the existence of a min and a max potential of $\mathcal{G}$.

If $\mathcal{G}$ has a max potential and a min potential then we could start Function *simple-increase* with the max potential. According to the discussion above, the function will return 'true' with a min-max potential p. ∎

Therefore, a min-max potential of $\mathcal{G}$ can be detected by first applying Function *simple-increase* to an arbitrary initial potential and then applying its dual counterpart Function *simple-decrease* to the resulting potential. $\mathcal{G}$ has a min-max potential if and only if Function *simple-increase* and Function *simple-decrease* both return 'true'.

This procedure constitutes a pseudo polynomial way to solve the min-max problem. However, cycles with a small weight sum, like e.g. $w(v_i, v_j) + w(v_j, v_i) = \epsilon \to 0$, in connection with large edge weights may lead to a large number of iterations. Therefore, we propose an improved algorithm by replacing functions *simple-increase* and *simple-decrease*.

As the new Function *path-increase* is based on shortest paths we use the notation $l(v_i, v_j)$ to describe the weight of the shortest path from node v_i to node v_j in a weighted digraph. If there is no path from v_i to v_j, then $l(v_i, v_j) = \infty$. Further, the min-reduction graph $\hat{\mathcal{G}}_{\min}$ is introduced:

Definition 3 (Min-Reduction Graph) *Assume a bipartite weighted digraph $\mathcal{G}(V, E, w)$, a start potential $p : V \to \mathcal{Q}$, and a predecessor function $b : V^+ \to V^- \cup \{v_0\}$ where $v_0 \notin V$ is an additional root node and $(b(v_i), v_i) \in E$ for all $v_i \in V^+$ with $b(v_i) \neq v_0$.*

A min-reduction graph of $\mathcal{G}$, p, and b is a graph $\hat{\mathcal{G}}_{\min}(\hat{V}_{\min}, \hat{E}_{\min}, \hat{w}_{\min})$ such that

$$
\begin{aligned}
\hat{V}_{\min} &= V^- \cup \{v_0\}, \\
\hat{E}_{\min} &= \{(v_i, v_j) \mid v_i, v_j \in \hat{V}_{\min} \textit{ and } \exists v_k \in V^+ \textit{ with } b(v_k) = v_i \textit{ and } (v_k, v_j) \in E\}, \\
\hat{w}_{\min}(v_i, v_j) &= \begin{cases} w(v_i, v_k) + w(v_k, v_j) & \textit{for } b(v_k) = v_i \neq v_0 \\ p(v_k) + w(v_k, v_j) & \textit{for } b(v_k) = v_i = v_0. \end{cases}
\end{aligned}
$$

A max-reduction graph $\hat{\mathcal{G}}_{\max}(\hat{V}_{\max}, \hat{E}_{\max}, \hat{w}_{\max})$ is defined in a similar fashion by exchanging V^- and V^+.

Informally, in the min-reduction graph each node $v_i \in V^+$ is attached to a node $v_j \in V^-$ or the root node v_0. A single source shortest path algorithm then guarantees that $p(v_j) = \min(p(v_i) + w(v_i, v_j) \mid (v_i, v_j) \in E)$ for all nodes $v_j \in V^-$. This idea is the basis for Function *path-increase* described in Table 2.

Corollary 5 *If and only if the bipartite min-max graph $\mathcal{G}$ has a min potential then Function* path-increase *returns 'true' and the generated potential p is a min potential.*

Proof: Assume a min-reduction graph during any iteration of the function and $v_j \in V^+$ with $b(v_j) \neq v_0$. Then, we have for any $(v_j, v_i) \in E$:

$$
l(v_0, v_i) \leq l(v_0, b_{old}(v_j)) + w(b_{old}(v_j), v_j) + w(v_j, v_i) \leq l(v_0, b_{new}(v_j)) + w(b_{new}(v_j), v_j) + w(v_j, v_i).
$$

```
Boolean Function path-increase(𝒢, p, 𝒢_t) {
        in 𝒢; inout p; out 𝒢_t;
        b(v) = v_0 for all v ∈ V^+;
loop:   generate the min-reduction graph Ĝ_min of 𝒢, p, and b;
        determine l(v_0, v_i) for all nodes v_i ∈ V^-;
        p(v_i) = l(v_0, v_i) for all v_i ∈ V^-;
        while (∃(v_i, v_j) ∈ E with v_j ∈ V^+ and p(v_j) < p(v_i) + w(v_i, v_j)) do {
                p(v_j) = p(v_i) + w(v_i, v_j); b(v_j) = v_i; }
        if (∄v ∈ V^+ for which p(v) has changed) {
                𝒢_t = subgraph of 𝒢 induced by all nodes v with p(v) < ∞;
                return (𝒢_t == 𝒢); }
        goto loop;
}
```

Table 2: Function *path-increase*

The case $b(v_i) = v_0$ can be omitted as no node $v_i \in V^+$ can obtain predecessor v_0 during the execution of the loop. Consequently, $p(v)$ cannot decrease for any $v \in V$ during the execution of Function *path-increase* with the exception of the first change of the initial potential for nodes from V^- if the initial potential was not feasible. On the other hand it is easy to see that $\sum_{v \in V^+} p(v)$ increases in every iteration but the last. Further note that the first min reduction graph is clearly free of any negative cycle. From the above follows that this must hold as well for all other min reduction graphs. As $l(v_0, v_i) \leq s$ or $l(v_0, v_i) = \infty$ for all $v_i \in V^-$ Function *path-increase* will terminate.

After the generation of each min reduction graph we have for all nodes $v_j \in V^+$

$$p(v_j) \leq \min_{(v_i, v_j) \in E} \{p(v_i) + w(v_i, v_j)\}.$$

If Function *path-increase* terminates with 'true' then the generated potential must therefore be a min potential.

Next assume that Functions *simple-increase* and *path-increase* are executed on the same graph $\mathcal{G}$ with the same initial potential p_{init}. We denote the potentials p of both functions by p_s and p_p, respectively. Then the edge selection in Function *simple-increase* can be done in a fashion such that:

1. $p_s(v) = p_p(v) = p_{init}(v)$ for all nodes $v \in V^+$ with $b(v) = v_0$,
2. $p_s(v) \leq p_p(v)$ for all other nodes in V.

If $\mathcal{G}$ has no min potential then Function *simple-increase* will produce $p_s(v) > a + s$ for some node $v \in V$. Therefore, we must have $p_p(v) = \infty$. ∎

Again for the detection of min-max potentials a dual Function *path-decrease* must be used as well.

3 The Dynamic Min-Max Problem

3.1 The Periodic Case

In this section we address the periodic min-max problem on dynamic graphs. To this end, first dynamic graphs are defined via static graphs as usual, see e.g. [5]. Then, Problem 1 is extended to dynamic graphs.

Definition 4 (Static Graph) *A (bipartite) static graph $\mathcal{G}_s(V = V^+ \cup V^-, E, w, d)$ is a bipartite weighted digraph with a weight function $w : E \to \mathcal{Z}$ and a distance function $d : E \to \mathcal{Z}_{\geq 0}$.*

Definition 5 (Dynamic Graph) *The dynamic graph corresponding to a given static graph $\mathcal{G}_s(V, E, w, d)$ is an infinite weighted bipartite graph $\mathcal{G}_d(V_d, E_d, w_d)$ where*

$$\begin{array}{lcl} V_d & = & \{v_i(k) \mid v_i \in V, k \in \mathcal{Z}_{\geq 0}\} \\ E_d & = & \{(v_i(k - d(v_i, v_j)), v_j(k)) \mid (v_i, v_j) \in E, k \in \mathcal{Z}, k \geq d(v_i, v_j)\} \\ w_d(v_i(k - d(v_i, v_j)), v_j(k)) & = & w(v_i, v_j) \ \textit{for all} \ (v_i(k - d(v_i, v_j)), v_j(k)) \in E_d \end{array}$$

Definition 6 (Periodic Min-Max Potential) *The periodic min-max potential $p_d : V_d \to \mathcal{Q}$ of a dynamic graph $\mathcal{G}_d(V_d, E_d, w_d)$ is a min-max potential p_d for all $k \geq K$ with $K = \max\{d(v_i, v_j) \mid (v_i, v_j) \in E\}$. Moreover, there is a period $\lambda \in \mathcal{Q}$ such that*

$$p_d(v_i(k + 1)) = p_d(v_i(k)) + \lambda \ \textit{for all} \ v_i(k) \in V_d$$

Problem 2 *Is there a periodic min-max potential for a dynamic graph $\mathcal{G}_d(V_d, E_d, w_d)$?*

In order to avoid dealing with infinite dynamic graphs, the cycle graph is introduced, see also [1].

Definition 7 (Periodic Cycle Graph) *For a static graph $\mathcal{G}_s(V, E, w, d)$ and a period $\lambda \in \mathcal{Q}$ the periodic cycle graph $\mathcal{G}_c(V, E, w_c)$ is the graph over V and E with the edge weight*

$$w_c(v_i, v_j) = w(v_i, v_j) - \lambda d(v_i, v_j) \ \textit{for all} \ (v_i, v_j) \in E.$$

Then the following corollary establishes a close relation between the periodic min-max problem of a dynamic graph and the min-max problem of the corresponding periodic cycle graph.

Corollary 6 *Assume a static graph $\mathcal{G}_s(V, E, w, d)$. Then, the following two statements are equivalent:*

- *The dynamic graph $\mathcal{G}_d$ corresponding to $\mathcal{G}_s$ has a periodic min-max potential p_d with period $\lambda \in \mathcal{Q}$.*
- *The periodic cycle graph $\mathcal{G}_c$ corresponding to $\mathcal{G}_s$ and λ has a min-max potential.*

Proof: A max potential of $\mathcal{G}_d$ requires for all $v_i(k) \in V_d^+$ and $k \geq K$ the correctness of the equation

$$p_d(v_i(k)) = \max_{(v_j(k-d(v_j,v_i)),v_i(k)) \in E_d} \{p_d(v_j(k - d(v_j, v_i))) + w_d(v_j(k - d(v_j, v_i)), v_i(k))\}.$$

Using the definition of a dynamic graph and the periodicity of p_d this equation can be transformed equivalently into

$$\begin{aligned} p_d(v_i(0)) &= p_d(v_i(k)) - k\lambda \\ &= \max_{(v_j(k-d(v_j,v_i)),v_i(k)) \in E_d} \{p_d(v_j(k - d(v_j, v_i))) + w_d(v_j(k - d(v_j, v_i)), v_i(k)) - k\lambda\} \\ &= \max_{(v_j,v_i) \in E} \{p_d(v_j(0)) + w_c(v_j, v_i)\} \end{aligned}$$

On the other hand, a periodic cycle graph with a max potential satisfies

$$p_c(v_i) = \max(p_c(v_j) + w_c(v_j, v_i) \mid (v_j, v_i) \in E) \text{ for all } v_i \in V^+.$$

With $p_d(v_i(0)) = p_c(v_i)$ for all $v_i \in V^+$ both conditions are equivalent. Similar arguments for all $v_i(k) \in V_d^-$ are used to conclude the proof. ■

The definitions and the above corollary are explained in the example shown in Figures 3 and 4.

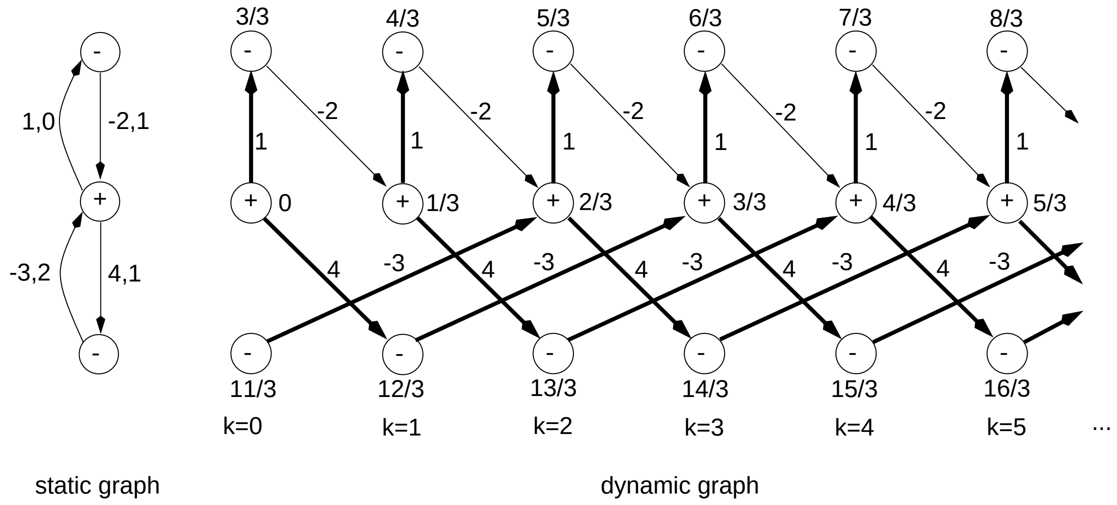

Figure 3: A static graph $\mathcal{G}_s$ and the corresponding dynamic graph $\mathcal{G}_d$. The edges of the static graph are labeled with $w(v_i, v_j)$, $d(v_i, v_j)$. The edges of the dynamic graph are labeled with $w(v_i(k), v_j(k))$ and to the nodes there are associated periodic min-max potentials $p_d(v_i(k))$. Tight edges are shown bold.

Corollary 6 states that the evaluation of the periodic cycle graph is sufficient for the computation of periodic min-max potentials. Further, if a dynamic graph $\mathcal{G}_d$ has a periodic min-max potential with period λ then the corresponding cycle graph $\mathcal{G}_c$ has at least one directed cycle C with $\sum_{(v_i,v_j) \in C} w_c(v_i, v_j) = 0$. Assuming $\sum_{(v_i,v_j) \in C} d(v_i, v_j) > 0$ this results in

$$\lambda = \frac{\sum_{(v_i,v_j) \in C} w(v_i, v_j)}{\sum_{(v_i,v_j) \in C} d(v_i, v_j)}. \tag{6}$$

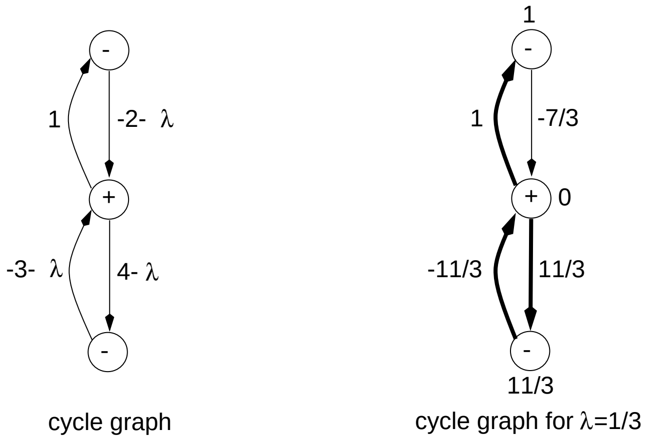

Figure 4: The periodic cycle graph $\mathcal{G}_c$ corresponding to the static graph $\mathcal{G}_s$ in Figure 3 and the periodic cycle graph for the period $\lambda = 1/3$. Min-max potentials $p_c(v_i)$ for $\lambda = 1/3$ are associated to the nodes.

Next, we introduce Function *lower-period* in Table 3 to determine the minimal period λ_{low} for which a periodic min-max potential may exist. This function is based on binary search, see also [14], [1], and includes a Function *increase* which may either stand for *simple-increase* or *path-increase* and a Function *critical-cycle* which is described below. Also, Function *lower-period* uses the following upper bound t for the sum of distances in any simple path in $\mathcal{G}_s$:

$$t = \sum_{v_j \in V} \left(\max_{(v_i,v_j) \in E} (|d(v_i, v_j)|) \right). \tag{7}$$

The purpose of Function critical-cycle($\mathcal{G}_s$, λ_l, λ_{low}) is to determine a period λ_{low} with $\lambda_l \le \lambda_{low} \le \lambda_u$ such that there exists a directed cycle C with $\sum_{(v_i,v_j)\in C} w_c(v_i, v_j) = 0$. One possibility can be described as follows:

- Determine the periodic cycle graph $\mathcal{G}_c$ corresponding to $\mathcal{G}_s$ and λ_l.
- Execute Function simple-increase($\mathcal{G}_c$, p, $\mathcal{G}_t$). The function returns 'false' and a subgraph $\mathcal{G}_t$.
- Determine a subset E'_t of edges of $\mathcal{G}_t$ which contains all tight edges to nodes in V_t^+ and the tightest edges to nodes in V_t^-. In particular, $E'_t = E'_1 \cup E'_2$ where $E'_1 = \{(v_i, v_j) \in E_t \mid p(v_j) = p(v_i) + w_c(v_i, v_j), v_j \in V_t^+\}$ and $E'_2 = \{(v_i, v_j) \in E_t \mid p(v_i) + w_c(v_i, v_j) - p(v_j) = min(p(v_k) + w_c(v_k, v_j) - p(v_j) \mid (v_k, v_j) \in E_t), v_j \in V_t^-\}$.
- Determine a cycle C using edges from E'_t only and calculate λ_{low} using Equation 6.

The correctness of Function *lower-period* is addressed in Corollary 7.

Corollary 7 *If Function* lower-period *returns 'false' then the dynamic graph $\mathcal{G}_d$ corresponding to $\mathcal{G}_s$ has no periodic min-max potential. Otherwise, $\mathcal{G}_d$ has min potentials for all $k \ge K$ and for all periods $\lambda \ge \lambda_{low}$ while there is no min-potential for all periods $\lambda < \lambda_{low}$.*

```
Boolean Function lower-period(𝒢_s, λ_low, p) {
        in 𝒢_s; out λ_l; inout p;
        determine s and t;
        λ_l = −s; λ_u = s;
        generate the periodic cycle graph 𝒢_c of 𝒢_s and λ_l;
        if  (increase(𝒢_c, p, 𝒢_t)) { λ_l = −∞; return 'true'; }
        generate the periodic cycle graph 𝒢_c of 𝒢_s and λ_u;
        if  (!increase(𝒢_c, p, 𝒢_t)) { return 'false'; }
loop    λ = (λ_u + λ_l)/2;
        generate the periodic cycle graph 𝒢_c of 𝒢_s and λ;
        if (!increase(𝒢_c, p, G_t)) { λ_l = λ; } else { λ_u = λ; }
        if  (λ_u − λ_l ≤ 1/t²) {
                critical-cycle (𝒢_s, λ_l, λ_low);
                return 'true';
        }
        goto loop;
}
```

Table 3: Function *lower-period*

Proof: Suppose that $\mathcal{G}_c$ has a min potential for $\lambda = -s$. Due to Corollary 2 there are min potentials for each $\lambda \geq -s$. Also note that for any cycle C with $\sum_{(v_i,v_j)\in C} d(v_i,v_j) > 0$ we already have $\sum_{(v_i,v_j)\in C}(w(v_i,v_j) - \lambda d(v_i,v_j)) \geq 0$ while other cycles are not affected by λ. Hence, any further reduction of λ will not influence the sign of the sum of weights in any cycle of $\mathcal{G}_c$. Therefore, $\mathcal{G}_c$ also has min potentials for all $\lambda < -s$.

The same arguments are also used to show that there is no min potential in $\mathcal{G}_c$ for any λ if $\mathcal{G}_c$ has no min potential for $\lambda = s$.

Function *lower-period* performs a binary search on λ. Assume two different periods λ_1 and λ_2 with different zero weight cycles in $\mathcal{G}_c$ which do not consist only of edges with distance 0. Then, the following lower bound for the difference between them holds:

$$|\lambda_1 - \lambda_2| = |\frac{\sum_{(v_i,v_j)\in C_1} w(v_i,v_j)}{\sum_{(v_i,v_j)\in C_1} d(v_i,v_j)} - \frac{\sum_{(v_i,v_j)\in C_2} w(v_i,v_j)}{\sum_{(v_i,v_j)\in C_2} d(v_i,v_j)}| \geq \frac{1}{t^2}.$$

Then, λ_{low} can be determined by use of the Function *critical-cycle*. ■

Similarly, a Function *upper-period* based on Function *decrease* is used to determine the maximal period λ_{up} for which a periodic min-max potential may exist. The combination of both functions yields an algorithm to determine whether there are periodic min-max potentials for a dynamic graph $\mathcal{G}_d$. The proof is a direct consequence of Corollary 7, its counterpart for Function *upper-period* and Theorem 1.

Theorem 2 *If either Function* lower-period *or Function* upper-period *return 'false' or if* $\lambda_{low} > \lambda_{up}$ *is produced, then there is no periodic min-max potential for the dynamic graph* $\mathcal{G}_d$ *corresponding to* $\mathcal{G}_s$. *Otherwise, there are periodic min-max potentials for all periods* $\lambda_{low} \leq \lambda \leq \lambda_{up}$.

In the following corollary we address the computational complexity for the presented method.

Corollary 8 *There is an algorithm which computes a periodic min-max potential in pseudo-polynomial time.*

Proof: The computational complexity for the whole method is identical to the complexity of Functions *lower-period* and *upper-period*. These functions perform a binary search while calling the Functions *increase/decrease* $O(log(st))$ times. ■

Finally, a result concerning the uniqueness of a period λ is derived.

Theorem 3 *If the static graph $\mathcal{G}_s$ corresponding to a dynamic graph $\mathcal{G}_d$ contains only edges with distance > 0 then $\mathcal{G}_d$ either has no periodic min-max potential or a min-max potential with a unique period.*

Proof: If $\mathcal{G}_c$ has a min-max potential there must be $\lambda = \frac{\sum_{(v_i,v_j)\in C} w(v_i,v_j)}{\sum_{(v_i,v_j)\in C} d(v_i,v_j)}$ as the denominator is positive. According to Theorem 2 any admissible periods are within an interval. This contradicts the minimal distance of two different periods as stated in the proof of Corollary 7. ■

3.2 The Quasi-Periodic Case

This section describes a generalization of the previous results. To this end, some of the definitions need to be reformulated and expanded. This also leads to a new problem.

Definition 8 (Quasi-Periodic Min-Max Potential) *The quasi-periodic min-max potential $p_d : V_d \to \mathcal{Q}$ of a dynamic graph $\mathcal{G}_d(V_d, E_d, w_d)$ is a min-max potential p_d for all $k \geq K$ with $K = \max\{d(v_i, v_j) \mid (v_i, v_j) \in E\}$. Moreover, there is a period function $\lambda : V \to \mathcal{Q}$ such that*

$$p_d(v_i(k+1)) = p_d(v_i(k)) + \lambda(v_i) \;\; \textit{for all} \;\; v_i(k) \in V_d$$

Problem 3 *Is there a quasi-periodic min-max potential for a dynamic graph $\mathcal{G}_d(V_d, E_d, w_d)$?*

As in the periodic case, a cycle graph is defined and its relation to a quasi-periodic min-max potential is established. Note that λ is always a period function in this section unless explicitly stated otherwise.

Definition 9 (Quasi-Periodic Cycle Graph) *For a static graph $\mathcal{G}_s(V, E, w, d)$ and a period function $\lambda : V \to \mathcal{Q}$ with $\lambda(v_i) \geq \lambda(v_j)$ for all $v_i \in V^+$ and v_i, v_j adjacent in $\mathcal{G}_s$ the quasi-periodic cycle graph $\mathcal{G}_c(V, E_c, w_c)$ is defined as follows:*

$$E_c = \{(v_i, v_j) \in E \mid \lambda(v_i) = \lambda(v_j)\}$$

$$w_c(v_i, v_j) = w(v_i, v_j) - \lambda(v_j) d(v_i, v_j) \;\; \textit{for all} \;\; (v_i, v_j) \in E_c.$$

The quasi-periodic cycle graph consists of not connected subgraphs where to each subgraph there is associated a period common to all nodes. As will be seen now, there is a one-to-one correspondence between the existence of a quasi-periodic min-max potential and the existence of min-max potentials in all subgraphs of the quasi-periodic cycle graph.

Corollary 9 *Assume a static graph $\mathcal{G}_s(V, E, w, d)$. Then, the following two statements are equivalent:*

- *The dynamic graph $\mathcal{G}_d$ corresponding to $\mathcal{G}_s$ has a quasi-periodic min-max potential p_d with the period function λ.*
- *The quasi-periodic cycle graph $\mathcal{G}_c$ corresponding to $\mathcal{G}_s$ and λ has a min-max potential.*

Proof: As in Corollary 6 a max potential of $\mathcal{G}_d$ requires for all $v_i(k) \in V_d^+$ and $k \geq K$ the correctness of the equation

$$p_d(v_i(k)) = \max_{(v_j(k-d(v_j,v_i)),v_i(k))\in E_d} \{p_d(v_j(k-d(v_j,v_i))) + w_d(v_j(k-d(v_j,v_i)),v_i(k))\}.$$

Using the definition of a dynamic graph and the periodicity of p_d we obtain the equivalent conditions

$$p_d(v_i(0)) = \max_{(v_j,v_i)\in E} \{p_d(v_j(0)) + w_c(v_j,v_i) + (k - d(v_j,v_i))(\lambda(v_j) - \lambda(v_i))\} \quad \forall k > K \tag{8}$$

1. $\mathcal{G}_d$ has a quasi-periodic min-max potential $\rightarrow$ $\mathcal{G}_c$ has a min-max potential.
   First assume that $\lambda(v_i) < \lambda(v_j)$. Also, the validity of Equation (8) *for all* $k > K$ prevents $p_d(v_i(0))$ from being finite.
   On the other hand if $\lambda(v_i) > \lambda(v_j)$ then edge (v_j, v_i) cannot affect Equation (8) for $k \rightarrow \infty$. Therefore, it need not be considered in Equation (8).
   In both cases (v_j, v_i) does not exist in $\mathcal{G}_c$.
   This results in
   $$p_d(v_i(0)) = \max_{(v_j,v_i)\in E_c} \{p_d(v_j(0)) + w_c(v_j,v_i)\} \quad \forall k > K$$
   which leads to a max potential $p_c(v_i) = p_d(v_i(0))$ for all $v_i \in V^+$.
2. $\mathcal{G}_c$ has a min-max potential $\rightarrow$ $\mathcal{G}_d$ has a quasi-periodic min-max potential.
   Suppose that a cycle graph is given with $p_c(v_i) = \max_{(v_j,v_i)\in E_c}\{p_c(v_j) + w_c(v_j,v_i)\}$ for all $v_i \in V_c$. Setting $p_d(v_i(0)) = p_c(v_i)$ for all $v_i \in E_c$, Equations (8) hold as $\lambda(v_i) \geq \lambda(v_j)$ and the third term in the max-expression becomes sufficiently negative for all edges in $E \backslash E_c$ and $k \rightarrow \infty$.

Similar arguments are used for all $v_i(k) \in V_d^-$. ∎

The definitions and the above corollary are explained in the example shown in Figures 5 and 6.

In order to simplify the following discussions we suppose that there is no directed cycle with a zero sum of distances in the given static graph $\mathcal{G}_s$, i.e.

$$\sum_{(v_i,v_j)\in C} d(v_i,v_j) > 0 \text{ for all directed cycles } C \text{ of } \mathcal{G}_s.$$

In the periodic case described in Section 3.1 the Functions *lower-period* and *upper-period* have been used to determine the single period λ for all nodes $v_i \in \mathcal{G}_s$. Following the result of Corollary 9 the solution in the quasi-periodic case consists of subgraphs of $\mathcal{G}_s$ with different periods.

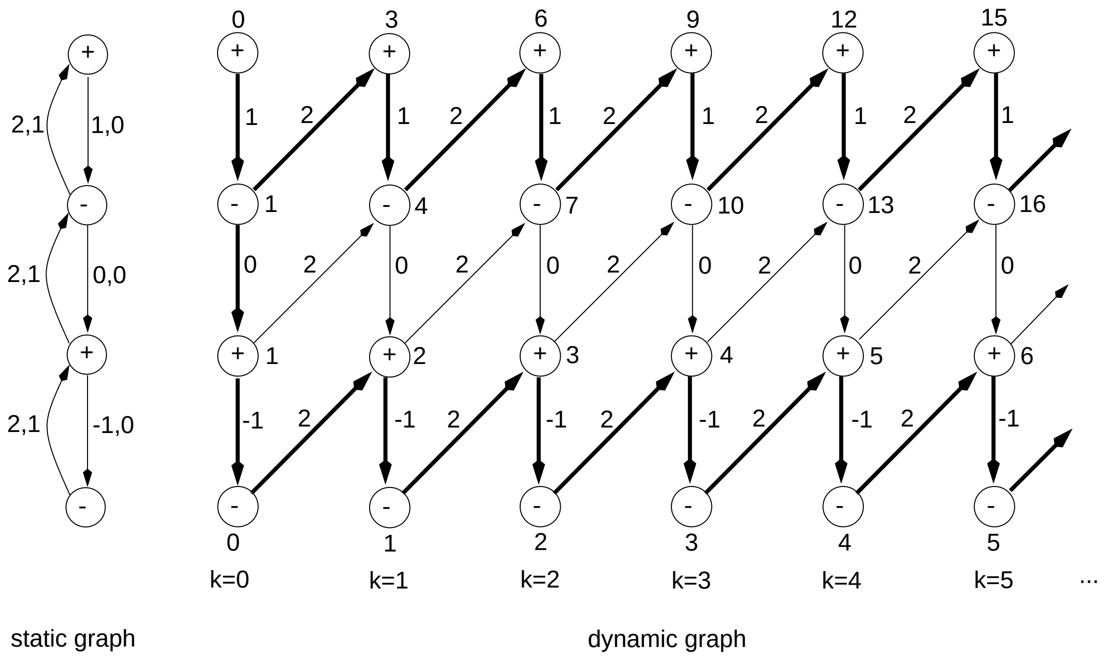

Figure 5: A static graph $\mathcal{G}_s$ and the corresponding dynamic graph $\mathcal{G}_d$. The edges of the static graph are labeled with $w(v_i, v_j)$, $d(v_i, v_j)$. The edges of the dynamic graph are labeled with $w(v_i(k), v_j(k))$ and to the nodes there are associated quasi-periodic min-max potentials $p_d(v_i(k))$. Tight edges are shown bold.

In other words, the quasi-periodic cycle graph $\mathcal{G}_c$ of $\mathcal{G}_s$ consists of unconnected subgraphs where each subgraph has a min-max potential and a period common to all nodes. This suggests an algorithm to determine the periods and subgraphs by iteratively pealing off subgraphs with decreasing periods from the static graph.

To this end, these subgraphs are first defined formally. This is done using the weighted bipartite graph $\mathcal{G}$ as used in the static min-max problem, see Section 2 and Definition 1,

Definition 10 (Dominating Subgraph) *A dominating subgraph $\mathcal{G}_t$ of a bipartite min-max graph $\mathcal{G}$ is a subgraph of $\mathcal{G}$ with the following properties:*

1. *There exists a min potential p of $\mathcal{G}$ which is a min-max potential of $\mathcal{G}_t$.*
2. *There are no edges (v_i, v_j) or (v_j, v_i) with $v_i \in V_t^-$ and $v_j \in (V^+ \backslash V_t^+)$.*

Based on this definition, a *maximal dominating subgraph* of a weighted digraph has the maximal number of nodes of all dominating subgraphs.

The following corollary provides results on one step of a procedure which determines the quasi-periodic min-max potential of a given static graph. It is shown that the concatenation of Functions *lower-period* and *decrease*

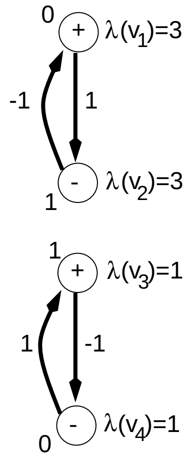

Figure 6: The quasi-periodic cycle graph $\mathcal{G}_c$ corresponding to the static graph $\mathcal{G}_s$ in Figure 5 and the periods $\lambda(v_1) = \lambda(v_2) = 3$ and $\lambda(v_3) = \lambda(v_4) = 1$. Min-max potentials $p_c(v_i)$ are associated to the nodes and tight edges are shown bold.

- peals off a subgraph of a given static graph,
- produces a period λ_{max} and a corresponding min-max potential for this subgraph and
- that the remaining static graph has a min potential for a period less than λ_{max}.

Corollary 10 *Given a static graph $\mathcal{G}_s$. After executing Function* lower-period($\mathcal{G}_s$, λ_{max}, p) *with initial potentials $p(v_i) = 0$ and subsequently executing Function* decrease($\mathcal{G}_c$, p, $\mathcal{G}_t$) *with the periodic cycle graph $\mathcal{G}_c$ corresponding to $\mathcal{G}_s$ and λ_{max}, the following properties hold:*

1. *$\mathcal{G}_t$ is a dominating subgraph of $\mathcal{G}_c$.*
2. *The application of Function* lower-period(($\mathcal{G}_s \backslash \mathcal{G}_t$), λ, p) *returns 'true' with $\lambda < \lambda_{max}$.*

Proof: Note that $\sum_{(v_i,v_j)\in C} d(v_i, v_j) > 0$ results in $\sum_{(v_i,v_j)\in C}(w(v_i, v_j) + s \cdot d(v_i, v_j)) \geq 0$ for all directed cycles C of $\mathcal{G}_s$. Therefore, Function *lower-period($\mathcal{G}_s$, λ_{max}, p)* cannot return 'false'.

Due to Corollary 7 $\mathcal{G}_d$ has no min potential for any $\lambda < \lambda_{max}$. Hence, application of Function *increase($\mathcal{G}_s$, p, $\mathcal{G}_{t'}$)* for $\lambda = \lambda_{max} - \epsilon$ with $\frac{1}{t^2} > \epsilon > 0$ must return 'false', that is $\mathcal{G}_{t'} \neq \mathcal{G}_c$. Then, there can be no edges (v_i, v_j) or (v_j, v_i) with $v_i \in V_{t'}^-$ and $v_j \in (V^+ \backslash V_{t'}^+)$. Also, application of Function *lower-period(($\mathcal{G}_{t'}$), λ, p)* will return 'true' with $\lambda < \lambda_{max}$. As the difference between any two simple paths with the same source and target node is finite, see proof of Corollary 7, we have $V \backslash V_{t'} = V_t$. Hence, there is a min-max potential for $\mathcal{G}_t$ and $\mathcal{G}_t$ is a dominating subgraph of $\mathcal{G}_c$. ■

Now, the complete algorithm for the calculation of the quasi-periodic min-max potential is given, see Table 4. Input to the Function *period($\mathcal{G}_s$, $\lambda()$)* is the given static graph $\mathcal{G}_s$ while its output

```
Boolean Function period(𝒢_s, λ()) {
        in 𝒢_s; out λ();
loop    p(v_i) = 0 for all v_i ∈ V_s;
        lower-period(𝒢_s, λ_max, p);
        generate the periodic cycle graph 𝒢_c of 𝒢_s and λ_max;
        if (decrease(𝒢_c, p, 𝒢_t)) {
                λ(v_i) = λ_max for all v_i ∈ V_t;
                return 'true'; }
        else {
                λ(v_i) = λ_max for all v_i ∈ V_t;
                𝒢_s = 𝒢_s\𝒢_t;
                goto loop;
        }
}
```

Table 4: Function *period*

is the resulting period function *lambda*. The corresponding min-max potentials can be either extracted during execution of the Function *period* or by using the proof of Corollary 9.

The following theorem states one of the main results of this paper.

Theorem 4 *Any dynamic graph $\mathcal{G}_d$ has a quasi-periodic min-max potential.*

Proof: The proof is a direct consequence of Corollary 10 as the potentials can be chosen such that $\min_{v_i \in V_t} p(v_i) > \max_{(v_i,v_j) \in E} |w(v_i, v_j)| + \max_{v_i \in V \setminus V_t} p(v_i)$. Therefore, both subgraphs will not affect each other. ∎

Finally, it remains to be shown whether the resulting period function λ and thus the partitioning of $\mathcal{G}_s$ in subgraphs with different periods is unique. To this end some more properties of a dominating subgraph and the quasi-periodic cycle graph are useful.

Corollary 11 *The maximal dominating subgraph of a bipartite min-max graph $\mathcal{G}$ is unique.*

Proof: Let $\mathcal{G}_{t1}$ and $\mathcal{G}_{t2}$ be two *maximal* dominating subgraphs of $\mathcal{G}$ with the corresponding potentials p_1 and p_2. Based on Definition 10 any node $v_i \in V_q = V_{t1} \setminus V_{t2}$ can only be incident with a node $v_j \in V \setminus V_q$ if

1. $v_i \in V_q^+$ and $v_j \in V \setminus V_{t2}$ or
2. $v_i \in V_q^-$ and $v_j \in V_{t1} \cap V_{t2}$.

Next, we consider the edge set $E' = (E_{p_2} \cap (V_q^+ \times V_q^-)) \cup (E_{p_1} \cap (V_q^- \times V_q^+))$ where E_{p_1} and E_{p_1} are the edges of the tightness graphs corresponding to $\mathcal{G}$ and p_1, p_2, respectively. Now, each node $v \in V$ belongs to a directed cycle C_v in E' as p_1 and p_2 are max and min potentials for the nodes in V_q, respectively.

Based on Definition 1 for p_1 and p_2, we obtain the following two inequalities:

$$p_1(v) \leq p_1(v) + \sum_{(v_i,v_j)\in C_v} w(v_i,v_j)$$

$$p_2(v) \geq p_2(v) + \sum_{(v_i,v_j)\in C_v} w(v_i,v_j).$$

Consequently, we have $\sum_{(v_i,v_j)\in C_v} w(v_i,v_j) = 0$ and each edge in C_v must be as well E_{p_1} as in E_{p_2}. Therefore, p_2 is a min-max potential for all nodes in V_q and $\mathcal{G}_{t1} \cup \mathcal{G}_{t2}$ must be a dominating subgraph with potential p_2. This contradicts that $\mathcal{G}_{t2}$ is a *maximal* dominating subgraph of $\mathcal{G}$. ∎

The next corollary describes the structure of a quasi-periodic min-max potential. In particular, it leads to the fact that any dynamic graph can be partitioned into a sequence of subgraphs which are dominating each other. The observation that these subgraphs are maximal dominating subgraphs provides the main result on the uniqueness of the period function associated to a dynamic graph.

Corollary 12 *Given a static graph $\mathcal{G}_s$. Then the following statements hold:*

1. *The maximal period $\lambda_{max} = \max\{\lambda(v_i) \mid v_i \in V_s\}$ is unique.*
2. *The subgraph $\mathcal{G}_t$ of $\mathcal{G}_c$ with $V_t = \{v_i \in V_s \mid \lambda(v_i) = \lambda_{max}\}$ is the* maximal *dominating subgraph of $\mathcal{G}_c$ where $\mathcal{G}_c$ is the periodic cycle graph corresponding to $\mathcal{G}_s$ and λ_{max}.*

Proof: If $\mathcal{G}_s$ has a quasi-periodic min-max potential then there exists a quasi-periodic cycle graph $\mathcal{G}_{qc}$ according to Definition 9 with a min-max potential, see Corollary 9. $\mathcal{G}_{qc}$ consists of not connected subgraphs. To each subgraph there is associated a period λ such that $\lambda(v_i) = \lambda$ for all nodes v_i of the subgraph. Now, let us construct the cycle graph $\mathcal{G}_c$ of $\mathcal{G}_s$ corresponding to λ_{max}. Then:

- The subgraph $\mathcal{G}_t$ in $\mathcal{G}_c$ containing nodes v_i with $\lambda(v_i) = \lambda_{max}$ in $\mathcal{G}_{qc}$ has a min-max potential in $\mathcal{G}_c$.
- There are no edges between nodes $v_i \in V_t^-$ and $v_j \in (V^+ \backslash V_t^+)$ as $\lambda(v_i) = \lambda_{max} > \lambda(v_j)$.
- $\mathcal{G}_c$ has a min potential. This can be seen by considering subgraphs of $\mathcal{G}_{qc}$ whose nodes v_i satisfy $\lambda(v_i) < \lambda_{max}$. If the weights of the edges are decreased from $w_{qc}(v_i,v_j) = w(v_i,v_j) - \lambda(v_j)d(v_i,v_j)$ to $w_c(v_i,v_j) = w(v_i,v_j) - \lambda_{max}d(v_i,v_j) < w_{qc}(v_i,v_j)$ then $\mathcal{G}_c \backslash \mathcal{G}_t$ is obtained. Following Corollary 2 and Theorem 3, this subgraph has a min potential.

Now, the two statements of the Corollary can be shown:

1. Let us suppose two distinct quasi-periodic min-max potentials of $\mathcal{G}_s$ with $\lambda_{max1} > \lambda_{max2}$ and the corresponding cycle graphs $\mathcal{G}_{c1}$ and $\mathcal{G}_{c2}$ with $w_{c1}(v_i,v_j) = w(v_i,v_j) - \lambda_{max1}d(v_i,v_j)$, $w_{c2}(v_i,v_j) = w(v_i,v_j) - \lambda_{max2}d(v_i,v_j)$ and therefore $w_{c1}(v_i,v_j) < w_{c2}(v_i,v_j)$. If the subgraph $\mathcal{G}_{t1}$ has a min-max potential then $\mathcal{G}_{c2}$ has no min potential and thus does not correspond to a quasi-periodic min-max potential, see Corollary 2 and Theorem 3. Therefore, λ_{max} is unique.

2. $\mathcal{G}_t$ is a dominating subgraph of $\mathcal{G}_c$ as there are no edges between nodes $v_i \in V_t^-$ and $v_j \in (V^+ \backslash V_t^+)$, $\mathcal{G}_t$ has a min-max potential and $\mathcal{G}_c$ has a min potential. $\mathcal{G}_t$ is a *maximal* dominating subgraph of $\mathcal{G}_c$. To proof this, let us suppose that a subgraph $\mathcal{G}_q$ of $\mathcal{G}_c \backslash \mathcal{G}_t$ could be moved to $\mathcal{G}_t$ and the resulting partition is still dominating. In this case, $\mathcal{G}_q$ must have a min-max potential with tight edges within $\mathcal{G}_q$ only, see the proof of Corollary 11. Note that the periods corresponding to nodes v_i in $\mathcal{G}_q$ satisfy $\lambda(v_i) < \lambda_{max}$. Therefore, in going back to the quasi-periodic cycle graph the weight of edges in $\mathcal{G}_q$ are increased and $\mathcal{G}_q$ has no min-max potential according to Corollary 2 and Theorem 3.

■

The following theorem summarizes the main result of the paper.

Theorem 5 *The period function λ of a given static graph $\mathcal{G}_s$ is unique.*

Proof: According to Corollary 12, the maximal period $\lambda_{max} = \max\{\lambda(v_i) \mid v_i \in V_s\}$ is unique. Moreover, using Corollary 11 and Corollary 12 it can be seen that the quasi-periodic cycle graph corresponding to a given static graph $\mathcal{G}_s$ has a unique subgraph $\mathcal{G}_t$ whose nodes v_i satisfy $\lambda(v_i) = \lambda_{max}$.

Because of the structure of the quasi-periodic cycle graph, the above arguments can now be recursively applied to the remaining static graph $\mathcal{G}_s \backslash \mathcal{G}_t$. Consequently, all subgraphs of the quasi-periodic cycle graph are unique and have a unique period. ■

4 Conclusion

The paper contains new results on a special class of weighted graphs, i.e. min-max graphs. These graphs model problems in different application areas, for example in circuit design and in the analysis and verification of communication protocols. These problems can be reduced to the determination of feasible potentials and period functions associated to the nodes.

Besides results on the existence and uniqueness of these potentials the present paper describes efficient pseudo-polynomial algorithms for their calculation. In particular, any dynamic graph $\mathcal{G}_d$ (whose static graph $\mathcal{G}_s$ has no directed cycle with a zero sum of distances) has a quasi-periodic min-max potential where to each node there is associated a private period. Moreover, the periods $\lambda(v_i)$ are unique for all $v_i \in \mathcal{G}_s$. In addition, there exists a pseudo-polynomial algorithm to compute these periods and min-max potentials. The case of equal periods, i.e. the case of periodic min-max potentials, can be considered to be a specialization thereof. These results are based on investigations and algorithms for weighted min-max graphs.

References

[1] Ahuja, R. K., Magnanti, T. L., and Orlin, J. *Network Flows.* Prentice Hall, 1993.

[2] Baccelli, F., Cohen, G., Olsder, G., and Quadrat, J.-P. *Synchronization and Linearity.* John Wiley, Sons, New York, 1992.

[3] Backes, W., Schwiegelshohn, U., and Thiele, L. Analysis of free schedule in periodic graphs. In *4th Annual ACM Symposium on Parallel Algorithms and Architectures* (San Diego, CA, USA, June 1992), pp. 333–342.

[4] Carlier, J., and Chretienne, P. Timed Petri Net Schedules. In *LNCS Advances in Petri Nets* (1988), vol. 340, Springer-Verlag, pp. 62–84.

[5] Cohen, E., and Megiddo, N. Strongly polynomial-time and NC algorithms for detecting cycles in dynamic graphs. *21st Annual ACM Symposium on Theory of Computing* (1989), 523–534.

[6] Cohen, G., Dubois, D., Quadrat, J. P., and Viot, M. A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing. *IEEE Transactions on Automatic Control AC-30, No. 3* (Mar. 1985), 210–220.

[7] Cunninghame-Green, R. Describing industrial processes and approximating their steady-state behaviour. *Opt. Res. Quart. 13* (1962), 95–100.

[8] Cunninghame-Green, R. Minimax algebra. In *Lecture Note 166 in Economics and Mathematical Systems*. Springer-Verlag, New York, 1979.

[9] Even, S., and Rajsbaum, S. The use of a synchronizer yields maximum computation rate in distributed networks. *22nd Annual ACM Symposium on Theory of Computing* (1990), 95–105.

[10] Gunawardena, J. Timing analysis of digital circuits and the theory of min-max functions. In *TAU'93, ACM International Workshop on Timing Issues in the Specification and Synthesis of Digital Systems* (Sept. 1993).

[11] Gunawardena, J. Min-max functions. Tech. Rep. to be published in Discrete Event Dynamic Systems, Department of Computer Science, Stanford University, Stanford, CA 94305, USA, Mar. 1994.

[12] Karp, R. A characterization of the minimum cycle mean in a digraph. *Discrete Mathematics 23* (1978), 309–311.

[13] Kosaraju, S. R., and Sullivan, G. F. Detecting cycles in dynamic graphs in polynomial time (preliminary version). In *20th Annual ACM Symposium on Theory of Computing* (1988), pp. 398–406.

[14] Lawler, E. Optimal cycles in doubly weighted directed linear graphs. In *Thèorie des Graphes* (1966), P. Rosenstiehl, Ed., pp. 209–213.

[15] McMillan, K., and Dill, D. Algorithms for interface timing verification. In *IEEE Int. Conference on Computer Design* (1992), pp. 48–51.

[16] Olsder, G. Eigenvalues of dynamic max–min systems. *Discrete Event Dynamic Systems: Theory and Applications* (1991), 1:177–207.

[17] Olsder, G. Analyse de systèmes min-max. Tech. Rep. 1904, Institute National de Recherche en Informatique et en Automatique, May 1993.

[18] Orlin, J. Some problems in dynamic and periodic graphs. In *Progress in Combinatorial Optimization* (Academic Press, Orlando, Florida, 1984), W.R. Pulleyblank, Ed., pp. 215–225.

[19] Ramamoorthy, C. Performance evaluation of asynchronous concurrent systems using Petri nets. *IEEE Transactions on Software Engineering* (1980), 440–449.

[20] Reiter, R. Scheduling parallel computations. *Journal of the Association for Computing Machinery 15, No. 4* (Oct. 1968), 590–599.

[21] Walkup, E., and Borriello, G. Interface timing verification with application to synthesis. In *IEEE/ACM Design Automation Conference* (1994), pp. 106–112.

[22] Yen, T., Ishii, A., Casavant, A., and Wolf, W. Efficient algorithms for interface timing verification. Tech. rep., Princeton University, June 1995.

[23] Zimmermann, U. Linear and combinatorial optimization in ordered algebraic structures. *Ann. Discrete Mathematics 10* (1981), 1–380.