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Dimensionality Reduction in Multiobjective Optimization: The Minimum Objective Subset Problem

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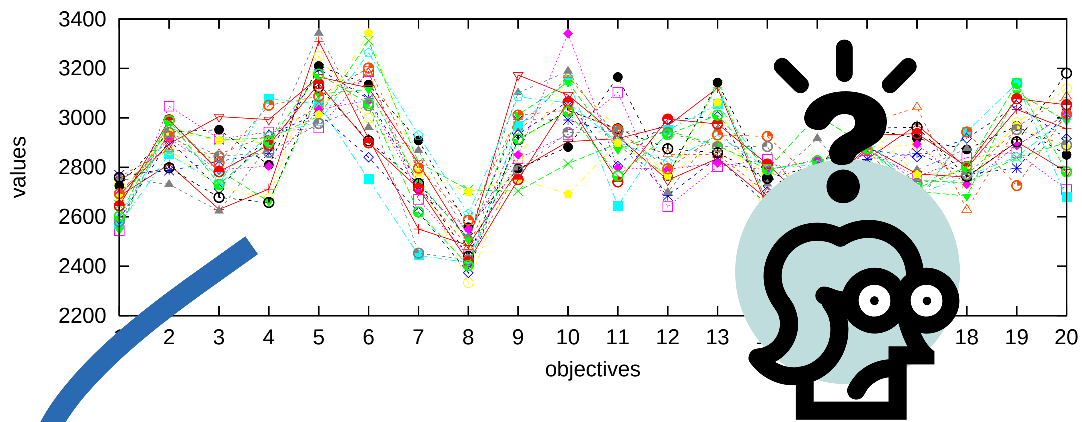
Motivation

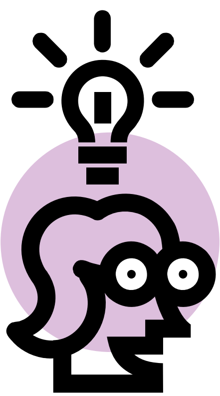
(Approximation of) efficient set

Multiobjective Problem

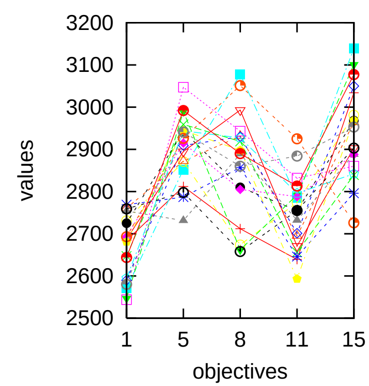
min. $\{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\}$
subject to $\mathbf{x} \in S$
and $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$

Generating method

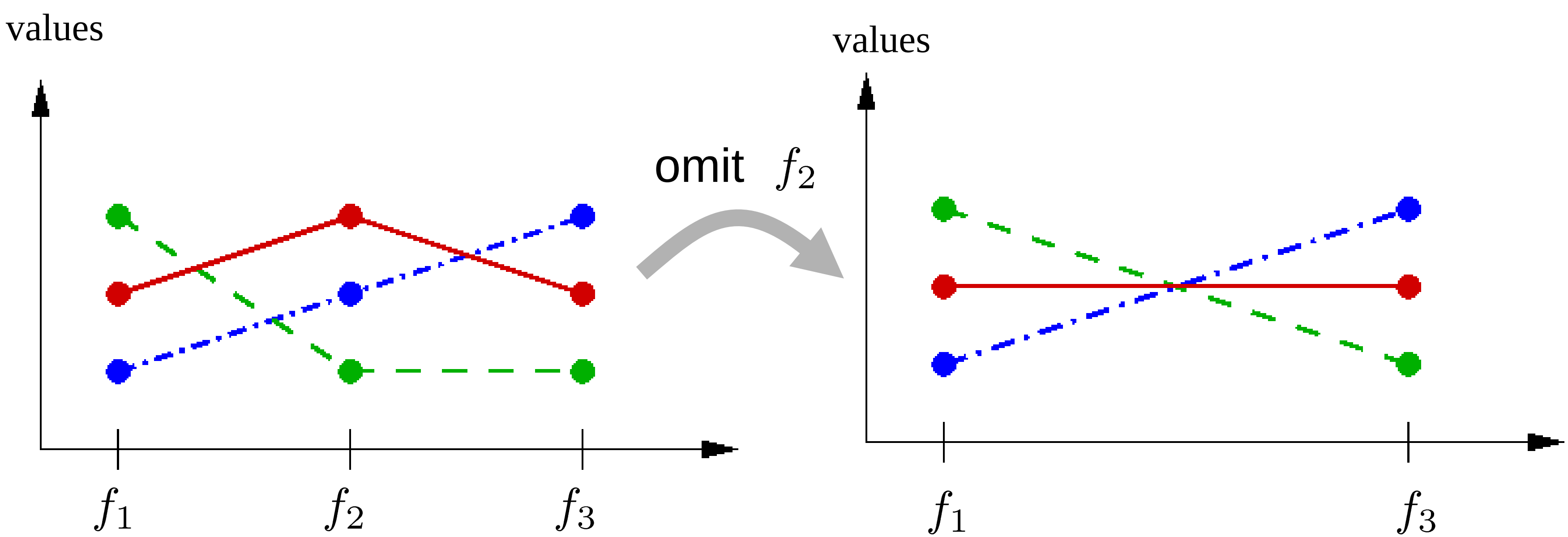




reduce number of objectives



$\Rightarrow$ assist decision maker



$a||b$ $b||c$ $a||c$

$a||b$ $b||c$ $a||c$

still the same relations

Key questions

- Objective reduction possible without changing the problem?
- How to compute a minimum objective set?
- Applicable to real problems?

Related Work

- Omitting redundant objectives:
  - Agrell (1997), Gal and Leberling (1977)
    - Not suitable for black-box optimization
- PCA based objective reduction:
  - Deb and Saxena (2005)
    - Cannot guarantee preservation of dominance structure
- Various conflict definitions:
  - Deb (2001); Tan et al. (2005)
    - conflict as a property of the problem itself
  - Purshouse and Fleming (2003):
    - *objective pairs* conflict if $\geq$ 2 solutions incomparable wrt the objective pair

Open Questions

- Conflicts between arbitrary objective sets
- Objective reduction with preservation of problem structure in a black-box scenario
- “Real“ problems

Key Contributions

- Generalization of Objective Conflicts
- The Minimum Objective Subset Problem
  - Exact and heuristic algorithms
- Objective reduction for selected problems

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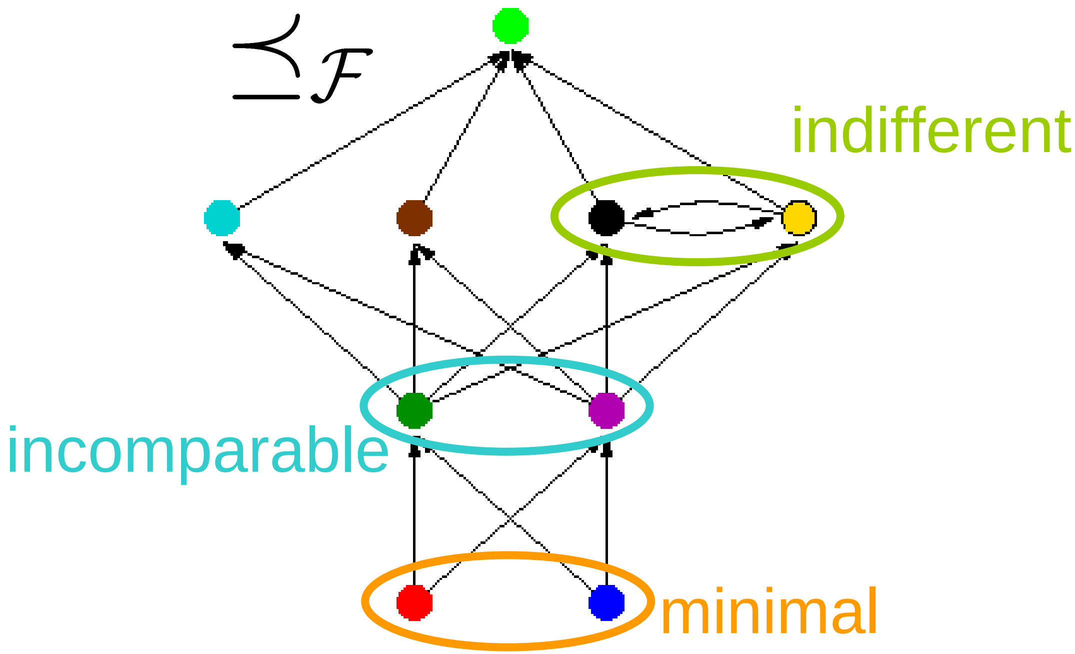
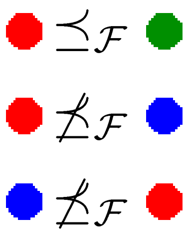
Relation Graphs and Dominance

- For a multiobjective problem, the question is to find the minimal elements of a given (pre)order $(X, \leq)$
- Here, we restrict to the weak dominance relation $\preceq_{\mathcal{F}}$

$$\preceq_{\mathcal{F}} := \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x}, \mathbf{y} \in X \wedge \forall f_i \in \mathcal{F} : f_i(\mathbf{x}) \leq f_i(\mathbf{y})\}$$

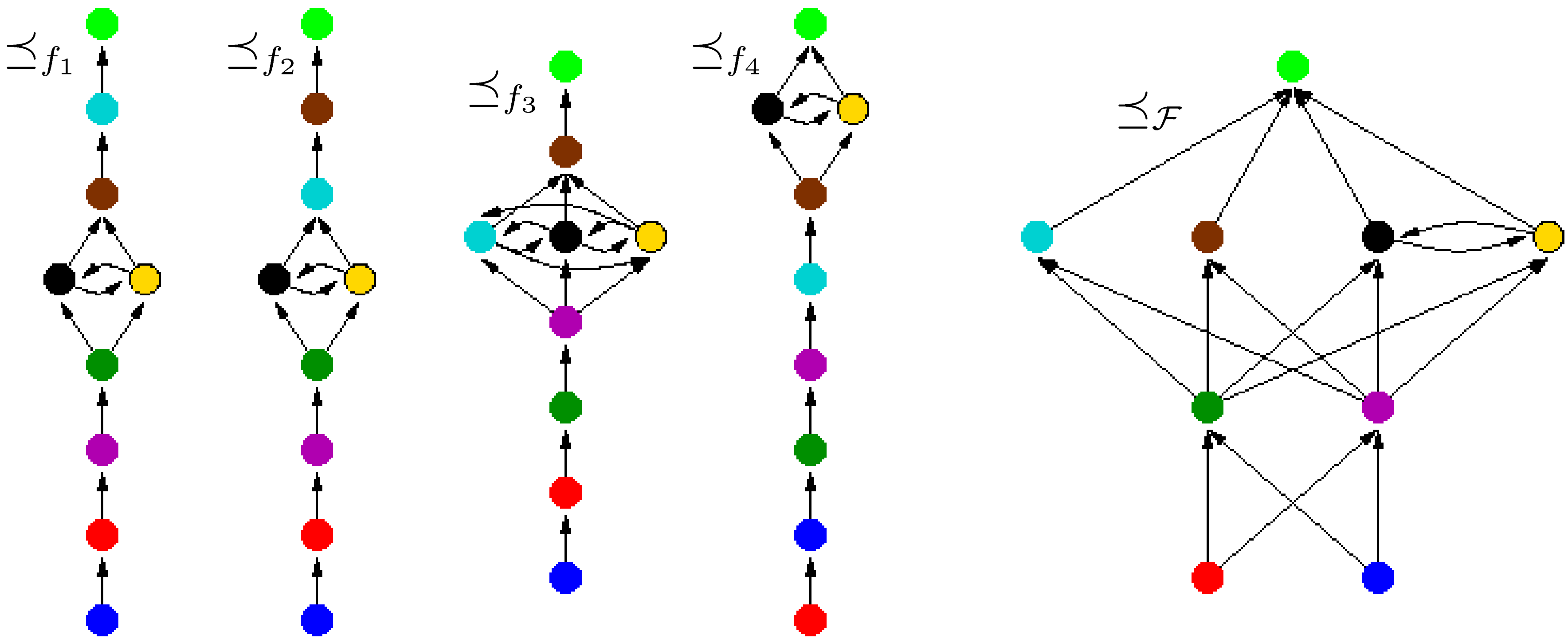




(reflexive and transitive edges are omitted)

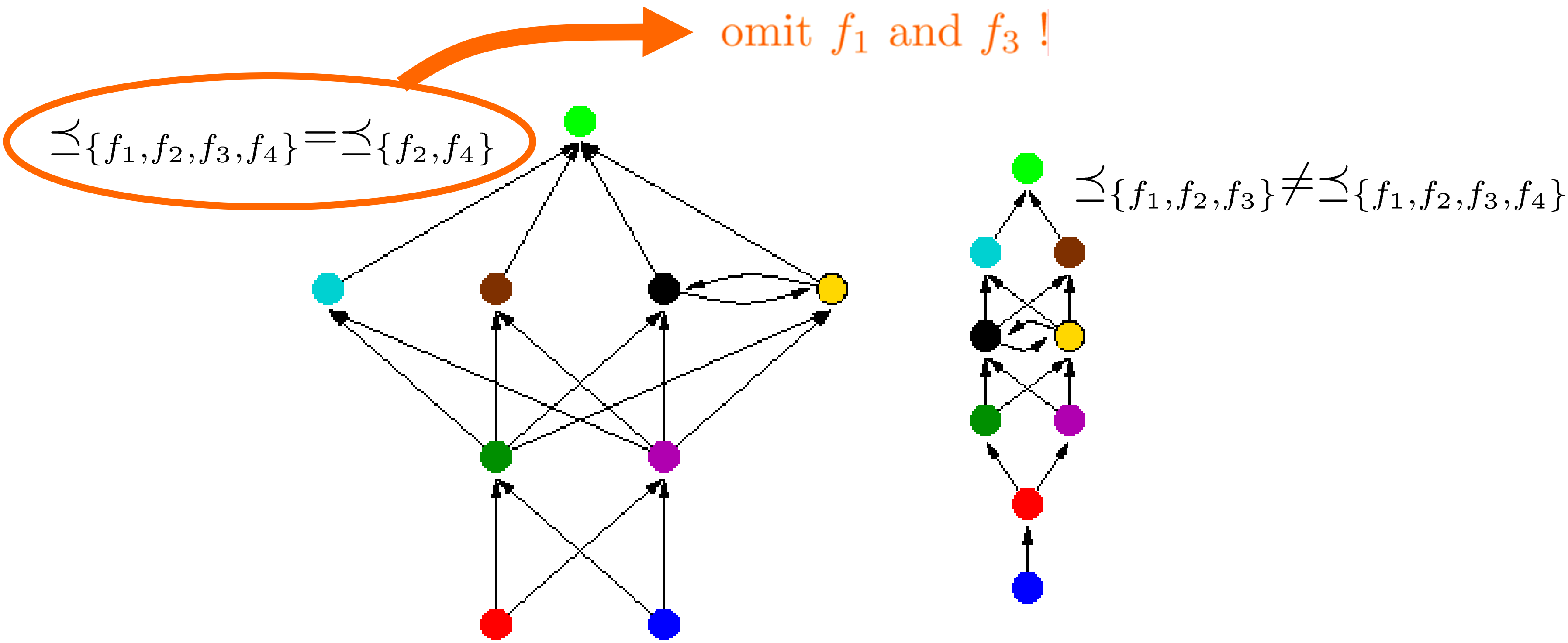
Intersection of Linear (Pre)Orders

- Single objectives induce linear (pre)orders $\preceq_{f_i}$
- Their intersection yields $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$
- Thus, the omission of objectives can only
  - make incomparable solution pairs comparable and
  - comparable solutions indifferent
  - add edges in relation graph



Objective conflicts

- Objective sets conflict if they induce different relations
  - **Definition:** $\mathcal{F}_1$ nonconflicting with $\mathcal{F}_2$ iff $\preceq_{\mathcal{F}_1} = \preceq_{\mathcal{F}_2}$
  - Omit objectives in $\mathcal{F} \setminus \mathcal{F}'$ if $\mathcal{F}' \subseteq \mathcal{F}$ is nonconflicting with $\mathcal{F}$ and preserve the dominance structure



Key Contributions

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The Minimum Objective Subset Problem

Minimum objective set

$\mathcal{F}' \subseteq \mathcal{F}$ is called minimum if $\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}$

and $\nexists \mathcal{F}'' \subseteq \mathcal{F} \wedge |\mathcal{F}''| < |\mathcal{F}'| : \preceq_{\mathcal{F}''} = \preceq_{\mathcal{F}}$

Minimum Objective Subset Problem (MOSS)

Given: Set A of solutions with weak dominance relations $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$ and $\preceq_{f_i} \subseteq A \times A$

Task: Compute a minimum objective set $\mathcal{F}' \subseteq \mathcal{F}$ with $\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}$

MOSS is NP-hard

- reduction from set cover problem (SCP)

Algorithms for the MOSS Problem

Exact algorithm

- Correctness proof
- Runtime: $O(|A|^2 \cdot k \cdot 2^k)$
- Worst case: $\Omega(|A|^2 \cdot 2^{k/3})$

$S := \emptyset$
for each pair $\mathbf{x}, \mathbf{y} \in A$ of solutions **do**
  $S_x := \{\,\{i\} \,|\, i \in \{1,\ldots,k\} \wedge \mathbf{x} \preceq_i \mathbf{y} \wedge \mathbf{y} \not\preceq_i \mathbf{x}\}$
  $S_y := \{\,\{i\} \,|\, i \in \{1,\ldots,k\} \wedge \mathbf{y} \preceq_i \mathbf{x} \wedge \mathbf{x} \not\preceq_i \mathbf{y}\}$
  $S_{xy} := S_x \sqcup S_y$ where
    $S_1 \sqcup S_2 := \{s_1 \cup s_2 \,|\, s_1 \in S_1 \wedge s_2 \in S_2$
      $\wedge (\nexists p_1 \in S_1, p_2 \in S_2 : p_1 \cup p_2 \subset s_1 \cup s_2)\}$
  if $S_{xy} = \emptyset$ **then** $S_{xy} := \{1,\ldots,k\}$
  $S := S \sqcup S_{xy}$
end for
Output a smallest set $s_{\min}$ in S

Simple greedy heuristic

- Correctness proof
- Runtime: $O(k \cdot |A|^2)$
- Best possible approximation ratio of $\Theta(\log |A|)$

$E := \preceq_{\mathcal{F}}^C$ where $\preceq_{\mathcal{F}}^C := (A \times A) \setminus \preceq_{\mathcal{F}}$
$I := \emptyset$
while $E \neq \emptyset$ **do**
  choose an $i \in (\{1,\ldots,k\} \setminus I)$
    such that $|\preceq_i^C \cap E|$ is maximal
  $E := E \setminus \preceq_i^C$
  $I := I \cup \{i\}$
end while

Key Contributions

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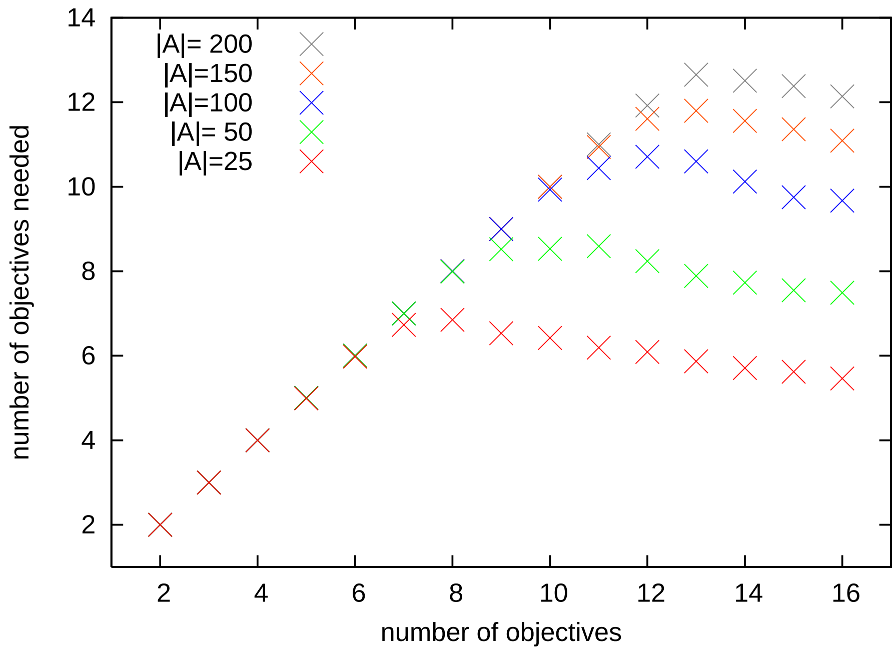
Objective Reduction for Selected Problems

- Solutions with randomly chosen objective values (i.e., random orders as $\preceq_{f_i}$):
  - Objective reduction possible?
  - Size of minimum set influenced by solution set size and number of objective?
  - Greedy vs. exact algorithm
- Realistic scenarios for test problems

Varying |A| and k for Random Orders

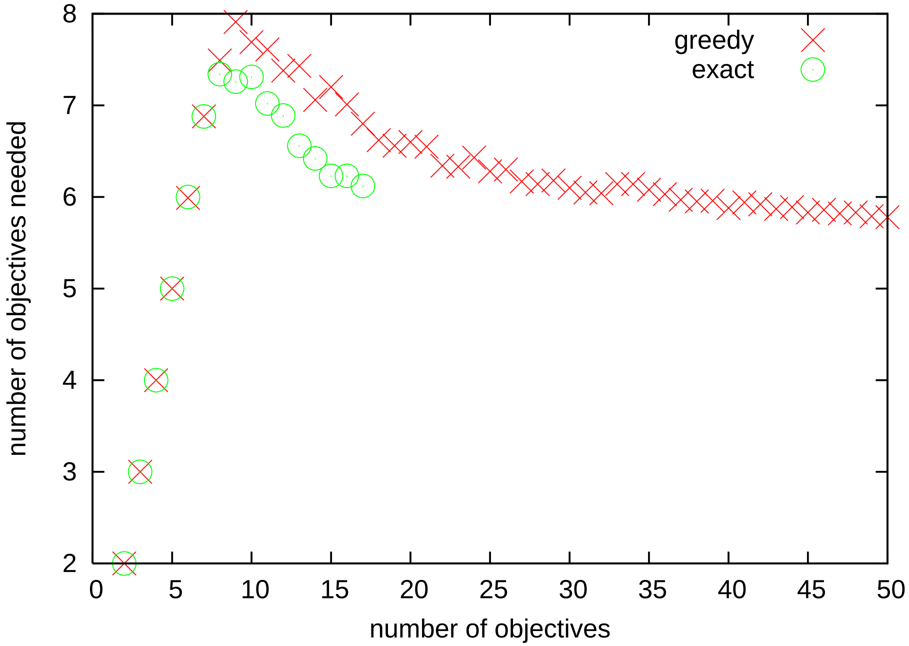
Various solution set sizes $|A|$ with random orders as $\preceq_{f_i}$



- The more objectives, the smaller the minimum sets
- The more solutions in A, the fewer objectives omissable

Greedy vs. Exact Algorithm for Random Orders

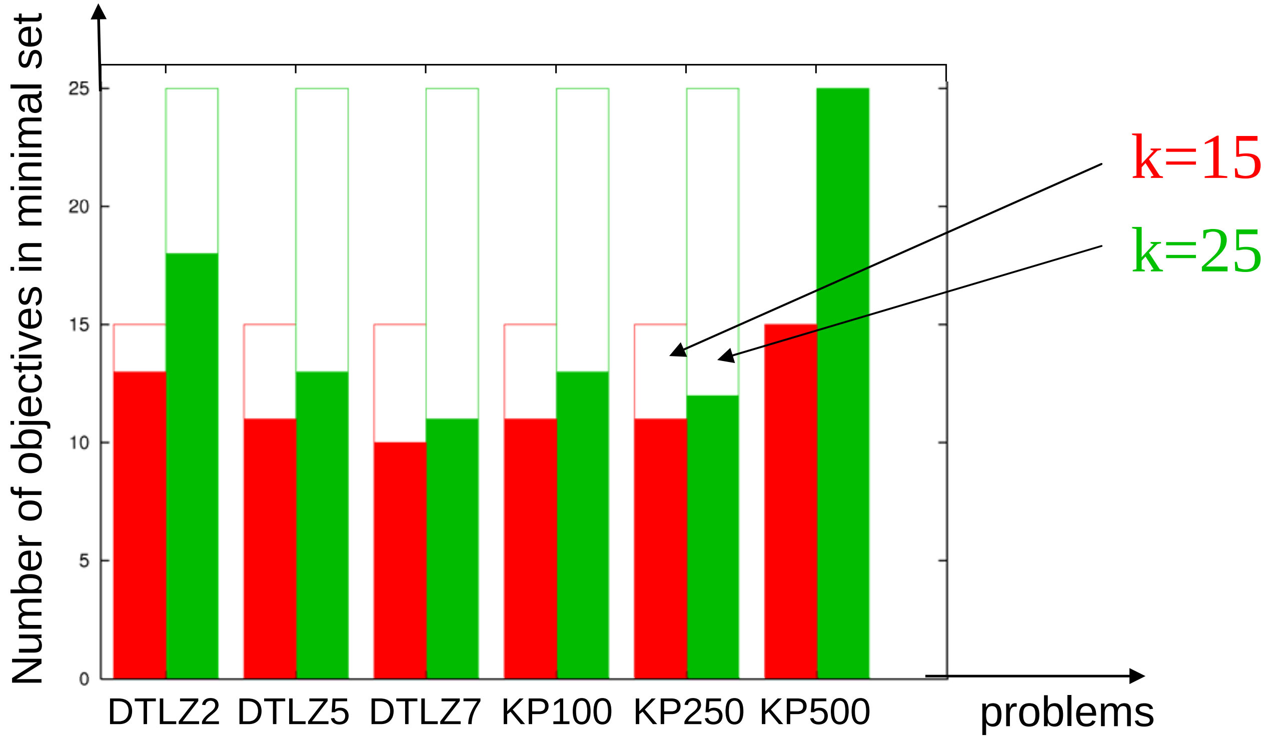
Heuristic vs. exact algorithm on random orders $\preceq_{f_i}$ with $|A| = 32$



- The greedy algorithm's objective sets are not too large
- Greedy algorithm has clearly lower running time:
  - can handle 50 objectives instead of $\leq$ 20 compared to exact algorithm within the same time

Realistic Scenarios for Test Problems

- Approximation of efficient set computed by evolutionary algorithm used as A
- $|A| = 200$ for $k = 15$ and $|A| = 300$ for $k = 25$



- Objective reduction of $\leq$ 50% possible for various test problems

Conclusions and Outlook

- Generalization of Objective Conflicts

- The MOSS Problem and algorithms
  - Often: preservation of structure too strict
  - Extension of approach to allow small changes in dominance structure: Brockhoff and Zitzler (2006)

- Method feasable for selected problems
  - Also for real world problems?
  - Method usable within generating methods?

Literature

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Parallel Coordinates Plot for Example