Word of Mouth: Rumor Dissemination in Social Networks

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Abstract. In this paper we examine the diffusion of competing rumors in social networks. Two players select a disjoint subset of nodes as initiators of the rumor propagation, seeking to maximize the number of persuaded nodes. We use concepts of game theory and location theory and model the selection of starting nodes for the rumors as a strategic game. We show that computing the optimal strategy for both the first and the second player is NP-complete, even in a most restricted model. Moreover we prove that determining an approximate solution for the first player is NP-complete as well. We analyze several heuristics and show that—counter-intuitively—being the first to decide is not always an advantage, namely there exist networks where the second player can convince more nodes than the first, regardless of the first player's decision.

1 Introduction

Rumors can spread astoundingly fast through social networks. Traditionally this happens by word of mouth, but with the emergence of the Internet and its possibilities new ways of rumor propagation are available. People write email, use instant messengers or publish their thoughts in a blog. Many factors influence the dissemination of rumors. It is especially important where in a network a rumor is initiated and how convincing it is. Furthermore the underlying network structure decides how fast the information can spread and how many people are reached. More generally, we can speak of diffusion of information in networks. The analysis of these diffusion processes can be useful for viral marketing, e.g. to target a few influential people to initiate marketing campaigns. A company may wish to distribute the rumor of a new product via the most influential individuals in popular social networks such as MySpace. A second company might want to introduce a competing product and has hence to select where to seed the information to be disseminated. In these scenarios it is of great interest what the expected number of persuaded nodes is, under the assumption that each competitor has a fixed budget available for its campaign.

The aim of this paper is to gain insights into the complexity of a model that captures the dissemination of competing rumors as a game where a number of players can choose different starting nodes in a graph to spread messages. The payoff of each player is the number of nodes that are convinced by the corresponding rumor. We focus on one crucial aspect of such a rumor game: the choice of a set of nodes that is particularly suitable for initiating the piece of information. We show that even for the most basic model, selecting these starting nodes is NP-hard for both the first and the second player. We analyze tree and d-dimensional grid topologies as well as general graphs with adapted concepts from facility location theory. Moreover, we examine heuristics for the selection of the seed nodes and demonstrate their weaknesses. We prove that contrary to our intuition there exist graphs where the first player cannot win the rumor game, i.e., the second player is always able to convince more nodes than the first player.

2 Related Work

Recently, viral marketing experienced much encouragement by studies [12] stating that traditional marketing techniques do no longer yield the desired effect. Furthermore [12, 15, 16] provide evidence that people do influence each other's decision to a considerable extent. The low cost of disseminating information via new communication channels on the Internet further increases the appeal of viral marketing campaigns. Thereupon algorithmic questions related to the spread of information have come under scrutiny. Richardson and Domingos [5] as well as Kleinberg et al. [13] were among the first to study the optimization problem of selecting the most influential nodes in a social network. They assume that an initial set of people can be convinced of some piece of information, e.g., the quality of a new product or a rumor. If these people later influence their friends' decisions recursively, a cascading effect takes place and the information is distributed widely in a network. They define the *Influence Maximization Problem*, which asks to find a k-node set for which the expected number of convinced nodes at the end of the diffusion process is maximized. The authors introduced various propagation models such as the linear threshold model and the independent cascade model. Moreover, they show that determining an optimal seeding set is NP-hard, and that a natural greedy hill-climbing strategy yields provable approximation guarantees. This line of research was extended by introducing a second competitor for the most far-ranging influence. Carnes et al. [3] study the strategies of a company that wishes to invade an existing market and persuade people to buy their product. This turns the problem into a Stackelberg game [23] where in the first player (leader) chooses a strategy in the first stage, which takes into account the likely reaction of the second players (followers). In the second stage, the followers choose their own strategies having observed the Stackelberg leader decision i.e., they react to the leader's strategy. Carnes et al. use models similar to the ones proposed in [13] and show that the second player faces an NP-hard problem if aiming at selecting an optimal strategy. Furthermore, the authors prove that a greedy hill-climbing algorithms leads to a $(1 - 1/e - \epsilon)$-approximation. Around the same time, Bharathi et al. [1] introduce roughly the same model for competing rumors and they also show that there exists an efficient approximation algorithm for the second player. Moreover they present an FPTAS for the single player problem on trees.

Whereas the application of information dissemination to viral marketing campaigns is relatively new, the classic subjects of *Competitive Location Theory* and *Voting Theory* provide concepts that are related and prove very useful in this paper. Location theory studies the question where to place facilities in order to minimize the distance to their future users. One of the earliest results stems from Hotelling [11], where he examines a competitive location problem in one dimension. He analyzes the establishment of ice-cream shops along a beach where customers buy their ice-cream at the nearest shop. *Voronoi Games* [4] study the same problem in two dimensions. In these games the location set is continuous, and the consumers are assumed to be uniformly distributed. Contrary to these assumptions the dissemination of information depends on the underlying network structure, i.e., there is a discrete set of possible "locations". Closest related to the spread of rumors is the competitive location model introduced by Hakimi [9]. Here, two competitors alternately choose locations for their facilities on a network. The author assumes that the first player knows of the existence of the second player and its budget, i.e., the leader can take the possible reactions of the follower into consideration. In turn, the follower has full knowledge of the leader's chosen positions and adapts its decision accordingly. Hakimi shows that finding the leader's and the follower's position on general graphs is NP-hard. Our model differs from Hakimi's two main aspects, namely he permits locating facilities on edges, and the placing of multiple users at nodes. Voting theory [10] introduces notions such as plurality solution, Condorcet solution or Simpsons solution describing the acceptance among a set of people, some of which we will use in our analysis.

A large body of research covers the dynamics of epidemics on networks, e.g., [2, 17–20] to name but a few. Many of these models are applicable to the diffusion of information for a single player, however, to the best of our knowledge no work exists on epidemics that fight each other.

3 Model and Notation

3.1 Propagation Models

The *Propagation Model* describes the dissemination of k competing rumors on an undirected graph $G(V, E)$. Initially, each node is in one of $k+1$ states. A node is in state $i \leq k$ if it believes rumor i, in state 0 if it has not heard any rumors yet. In the first step all nodes apart from the nodes in state 0, send a message containing rumor i to their neighbors, informing them about their rumor. Now, all nodes in state 0 that received one or more messages decide which rumor they believe (if any), i.e. they change their state to i if they decided to accept rumor i, or remain in state 0, or adopt state ∞ if they reject all rumors. Nodes in state $i \in \{1 \dots k\}$ spread the rumor by forwarding a message to their neighbors. These steps are repeated recursively until no messages are transmitted any more. Observe that in this model each node transmits at most once and no node ever changes its first decision.

Depending on the process of reaching a decision after receiving one or several messages the diffusion of the rumors differs. In this paper we mostly consider the *basic model* where each node trusts the first rumor it encounters unless two or more different rumors arrive at the same time in which case the node chooses state ∞, i.e., it refuses to decide and ignores all further messages.

This model can easily be extended by varying the decision process. E.g., rumor i could be accepted and forwarded to the neighbors with probability $P_{rumor_i} = \frac{\#\text{messages rumor}_i}{\#\text{messages}}$. Thereby the decision depends on the number of messages containing rumor i versus the total number of messages received in this time slot. Moreover, edges could be oriented and a persuasiveness value could be assigned to each rumor influencing the decision. A more complex model such as the *linear threshold model* or the *independent cascade model* could be implemented. Note that our basic model is a special case of the independent cascade model. The threshold model has been introduced by Granovetter [8] and Schelling [21], who were among the first to define a model that handles the propagation of information in networks. In this model, a node u forwards a rumor i to all neighbors if the accumulated persuasiveness of the received messages i exceeds a threshold, $\sum_{m_i} psv_u(m_i) \geq t$. The independent cascade model has been proposed in the context of marketing by Goldenberg, Libai and Muller [6]. Here, a node u is given one opportunity to propagate rumor i to neighbor v with probability $p_{u,v}$. Thereafter no further attempts of node u to convince node v take place. Kempe et al. [14] show that these two models can be generalized further and ultimately are equivalent.

3.2 Strategic Rumor Game

Consider two players p_1, p_2 and a graph $G(V, E)$. Player p_1 selects a subset $V_1 \subset V$ of nodes corresponding to the set of nodes initiating rumor 1. Subsequently, p_2 selects the seeds for rumor 2, a set $V_2 \subset V$, where $V_1 \cap V_2 = \emptyset$. The rumors then propagate through the graph as specified by the propagation model. The payoff for player p_i is calculated when the propagation has terminated and equals the number of nodes that believe $rumor_i$. This model can be extended to multiple players, where each players' strategy consists of a disjoint set of nodes to initiate their rumors.

Observe that this game is related to the classic subject of competitive location theory and the equilibrium analysis of voting processes. In order to analyze our rumor game in different topologies we therefore introduce the notions *Distance Score* and *Condorcet Node.*

Definition 1 *For any two nodes $v_i, v_j \in V$ the number of nodes that are closer to v_i than to v_j is designated as the* distance score, $DS_i(j) = |\{v \in V : d(v, v_i) < d(v, v_j)\}|$. *A node $v_j \in V$ is called a* Condorcet node *if $DS_i(j) \leq |V|/2$ for every $v_i \in V \setminus \{v_j\}$.*

Thus a node $v_j \in V$ is called a Condorcet node if no more than one half of the nodes accept a rumor from any other node in the graph. Note that this definition differs from the original definition of a *Condorcet Point* that can be anywhere on the graph, including edges.

4 Analysis

Location theory studies the optimal distribution of facilities such that the distance to the users is minimized. In our basic model, we consider a very similar problem. Instead of two facility providers two rumors compete for users. Hakimi et al. [9] examine the facility location problem in a weighted graph, i.e., each edge is assigned a length value. The facilities are located at nodes or edges, the users are located at nodes only and multiple users are allowed per node. We adjust these concepts to our model where only one user is located at each node and the edge lengths are restricted to 1. Furthermore, the rumors cannot start on edges, i.e., the available locations are confined to the nodes.

The $(r|p)$-medianoid problem in location theory asks to locate r new facilities in the graph which compete with p existing facilities for reaching more users. Whereas the $(r|p)$-centroid problem examines how to place the p facilities when knowing that r facilities are located afterwards by a second player. We adapt these two terms for the problems faced by player 1 and player 2 in the rumor game.

Definition 2 *Player 1 solves the* $(r|p)$-centroid *problem of a graph by selecting p nodes to initiate rumor 1 ensuring that the number of nodes convinced by rumor 1 is maximized when player 1 knows that player 2 will choose r nodes.*

Definition 3 *Player 2 solves the* $(r|p)$-medianoid *problem of a graph by selecting r nodes to initiate rumor 2 ensuring that the number of nodes convinced by rumor 2 is maximized when player 1 has chosen p nodes already.*

The locational centroid and medianoid problems have been shown to be NP-complete in [9]. Our rumor game using the basic model is a restricted special case of the general facility location problem. In the following paragraphs we will prove that the computation of optimal solutions in the rumor game is of the same difficulty. To this end we need some additional notation. Let $D_G(v, Z) = \min\{d(v, z)|z \in Z\}$ for a subset $Z \subset V$, where $d(v, z)$ describes the length of a shortest path from v to z in G. Thus $D_G(v, Z)$ designates the lengh of the shortest path from node v to a node $z \in Z$. Let X_p be the set of the p nodes

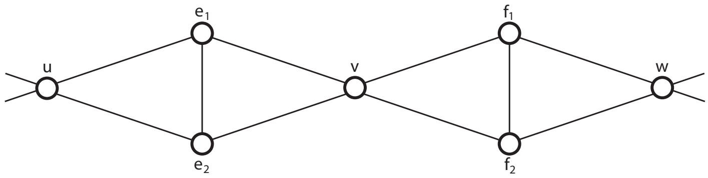

Fig. 1. Diamond structure used for the reduction of the centroid problem.

chosen by player 1 and Y_r the set of the r nodes selected by player 2. The set of nodes that are closer to a rumor published by Y_r than to the ones published by X_p is $V(Y_r|X_p) = \{v \in V | D(v, Y_r) < D_G(v, X_p)\}$. This allows us to define the part of the graph controlled by rumors placed at Y_r as $W(Y_r|X_p) = |V(Y_r|X_p)|$.

4.1 Complexity of the Centroid Problem

We demonstrate how the $(1|p)$-centroid problem can be reduced to from Vertex Cover.

Theorem 4 *The problem of finding an* $(1|p)$*-centroid of a graph is NP-hard.*

Proof. We prove this theorem by reducing the *Vertex Cover (VC)* problem to the $(1|p)$-centroid problem. An instance of the VC problem is a graph $G(V, E)$ and an integer $p < |V|$. We have to determine whether there is a subset $V' \subset V$ with $|V'| \leq p$ such that each edge $e \in E$ has at least one end node in V'.

Given an instance of the VC problem, we construct a graph $\bar{G}(\bar{V}, \bar{E})$ from G by replacing each edge $e_i = (u, v)$ in G by the diamond structure shown in Figure 1. Let $Y_1(X_p)$ be the node chosen by player 2 when player 1 has selected the nodes X_p. We prove our theorem by showing that there exists a set X_p of p nodes on $\bar{G}$ such that $W(Y_1(X_p)|X_p) \leq 2$ for every node $Y_1(X_p)$ on $\bar{G}$, if and only if the VC problem has a solution.

Assume V' is a solution to the VC problem in G and $|V'| = p$. Let $X_p = V'$ on $\bar{G}$. Then for any diamond joining u and v in $\bar{G}$, either u or v belongs to $V' = X_p$. It is easy to see that in this case $W(Y_1(X_p)|X_p) \leq 2$ for every node $Y_r(X_p)$ in $\bar{G}$. On the other hand suppose the set of p nodes X_p on $\bar{G}$ satisfies the requirement $W(Y_1(X_p)|X_p) \leq 2$ for every choice of node $Y_1(X_p)$ on $\bar{G}$. If on each diamond of $\bar{G}$ there exists at least one node of X_p, then we can move this node to u or $v \in V' \subset V$. It follows that each diamond has either u or v in V' and therefore V' would provide a solution to the VC problem in G. What can we say about diamonds in $\bar{G}$ joining u and v on which no node of X_p lies? Without loss of generality we may state that there has to be an adjacent diamond with at least one node of X_p, otherwise $W(Y_1(X_p)|X_p) \leq 2$ is violated. No matter whether w, f_1 or f_2 is in X_p, player 2 can select v yielding $W(Y_1(X_p)|X_p) \geq 3$. Consequently, player 1 has to choose at least one node on each diamond and the claim follows. □

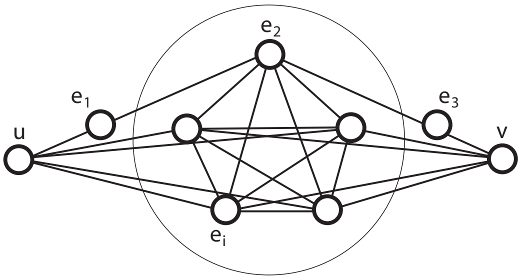

Fig. 2. Graph $\bar{G}$ used for the reduction of the approximation of the centroid problem.

Note that for trees the(1|1)-centroid is always on a node in the facility location context [22]. Hence the algorithm proposed by Goldman [7] can be used to find an(1|1)-centroid on trees in time $O(n)$.

Intriguingly, even finding an approximate solution to the $(1|p)$-centroid problem is NP-hard. We define X_p^α to be an α-approximate $(1|p)$-centroid if for any $1 < \alpha \in o(n)$ it holds that $W(Y_1^{OPT}(X_p^\alpha)|X_p^\alpha) \leq \alpha W(Y_1^{OPT}(X_p^{OPT})|X_p^{OPT}$.

Theorem 5 *Computing an α-approximation of the $(1|p)$-centroid problem is NP-hard.*

Proof. This proof uses a reduction from Vertex Cover again and follows the previous proof closely. Given an instance of the VC problem, we construct a graph $\bar{G}(\bar{V}, \bar{E})$ from G by replacing each edge $e_i = (u, v)$ in G by another diamond structure shown in Figure 2. Instead of adding two nodes and five edges for every edge (u, v), we introduce a clique of $4\alpha - 2$ nodes and connect u and v to each node of the clique. Moreover we insert one node on each of the edges from u, v to one designated node of the clique. In a first step we show that $W(Y_1(X_p)|X_p) \leq 4\alpha$ for every node player 2 might pick as Y_1 if and only if VC has a solution.

Assume V' is a solution to the VC problem in G and $|V'| = p$. Let $X_p = V'$ on $\bar{G}$. Then for any diamond joining u and v in $\bar{G}$, either u or v belongs to $V' = X_p$. It is easy to see that in this case $W(Y_1(X_p)|X_p) \leq 4 \leq 4\alpha$ for every node $Y_r(X_p)$ in $\bar{G}$. On the other hand suppose X_p satisfies $W(Y_1(X_p)|X_p) \leq 4\alpha$ for every choice of node $Y_1(X_p)$ on $\bar{G}$. If on each diamond of $\bar{G}$ there exists at least one node of X_p, then we can move this node to u or $v \in V' \subset V$. It follows that each diamond has either u or v in V' and therefore V' would provide a solution to the VC problem in G.

Suppose there is a diamond without a node in X_p. In this case, it is easy to see that if $\min\{D(u, X_p), D(v, X_p)\}$ exceeds one, $W(Y_1(X_p)|X_p) \geq 4\alpha + 2$. Hence we may assume that $0 < \min\{D(u, X_p), D(v, X_p)\} \leq 1$ and we can state without loss of generality that there has to be an adjacent diamond with at least one node of X_p in distance 1 to u. No matter which of the suitable

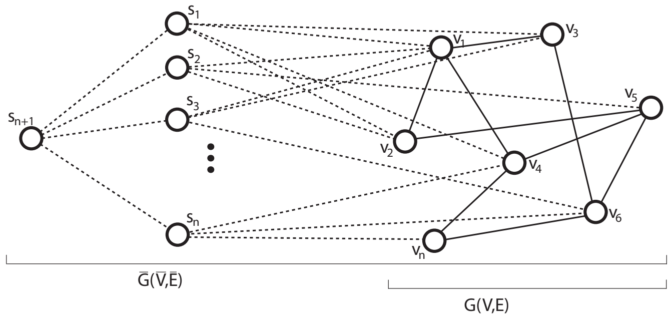

Fig. 3. Graph $\bar{G}$ used in the reduction of the medianoid problem.

nodes is in X_p, player 2 can select u yielding $W(Y_1(X_p)|X_p) \geq 4\alpha + 1$. Consequently, player 1 has to add another node on this diamond to X_p to avoid a violation of our presumption. Thus we can easily construct a VC out of X_p. Moreover, we can prove using similar arguments that X_p exists on $\bar{G}$ such that the condition that $W(Y_1(X_p)|X_p) \leq 4$ holds for every node $Y_1(X_p)$ on $\bar{G}$ if and only if VC on G has a solution. Consequently it must hold that $W(Y_1(X_p)|X_p) \leq \alpha W(Y_1^{OPT}(X_p^{OPT})|X_p^{OPT}) \leq 4\alpha$ and the statement of the theorem follows. □

4.2 Complexity of the Medianoid Problem

The second player has more information than the first player, however, determining the optimal set of seeding nodes for player 2 is in the same complexity class. We prove this by a reduction of the dominating set problem to the $(r|X_1)$-medianoid problem.

Theorem 6 *The problem of finding an* $(r|X_1)$*-medianoid of a graph is NP-hard.*

Proof. Consider an instance of the NP-complete *Dominating Set (DS)* problem, defined by a graph $G(V, E)$ and an integer $r < n$, where $n = |V|$. The answer to this problem states whether there exists a set $V' \subset V$ such that $|V'| \leq r$ and $D_G(v, V') \leq 1$ for all $v \in V$. We construct a graph $\bar{G}(\bar{V}, \bar{E})$ with node set $\bar{V} = V \cup S$, where S consists of $n+1$ nodes. Let the nodes in V be numbered from $v_1, \ldots, v_n$ and the nodes in S from $s_1, \ldots, s_{n+1}$. For each node s_i, $i \in \{1, \ldots, n\}$, we add an edge to s_{n+1}, an edge to v_i as well as an edge to every neighbor of v_i, compare Figure 3. Thus the edge set is $\bar{E} = E \cup E_s$, where $E_s = \{(s_i, v_i)|s \in S, v \in V\} \cup \{(v_{n+1}, v)|v \in S \setminus \{v_{n+1}\}\} \cup \{(s_i, v_j)|(v_i, v_j) \in E\}$. Let player 1 choose s_{n+1}. We show now that there exist r nodes in $\bar{G}$ composing Y_r such that $W(Y_r|s_{n+1}) = |V| + r$, if and only if the DS problem has a solution in G.

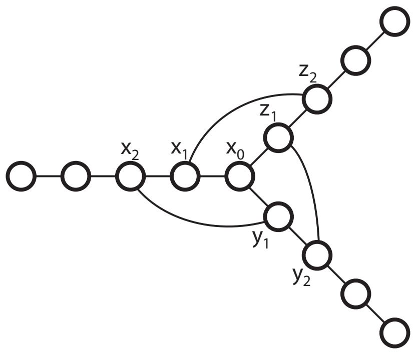

Fig. 4. Example of a graph where the first player never wins.

Assume the DS problems has a solution in G. In this case there exists $V' \subset V$ with $|V'| = r$ such that $D_G(v, V') \leq 1$ for all $v \in V$. Let Y_r contain the nodes in S corresponding to V', i.e., $Y_r = \{s_i | v_i \in V'\}$. It follows that $W(Y_r|s_{n+1}) = |V| + r$, because $\forall v \in V \;\; D(v, Y_r) = 1 < d(v, s_{n+1}) = 2$. Suppose Y_r is such that $W(Y_r|s_{n+1}) = |V| + r$. For all nodes $s_i \in Y_r$ it holds that $W(s_i|s_{n+1}) \leq W(v_j|s_{n+1})$, if s_i and v_j are neighbors. This follows from the fact that on every path from a node $v \in V$ to s_{n+1} in $\bar{G}$ there is a node s_i, $i < n+1$. By removing s_i from Y_r and adding its neighbor $v_i \in V$ to Y_r we maintain $\forall v \in V \;\; D(v, Y_r) = 1 < d(v, s_{n+1}) = 2$. We repeat these steps for all nodes $s_i \in S \cap Y_r$ yielding $Y_r \subset V$. Clearly, $W(Y_r|s_{n+1}) = |V|$, letting us state for all $v \in V, D(v, Y_r) < d(v, s_{n+1}) = 2$. Thus this adapted set Y_r is a solution to DS. □

Observe that the hill-climbing algorithms proposed in [3] can be adapted to provide $(1 - 1/e - \epsilon)$-approximations of the medianoid problem in polynomial time.

4.3 Advantage of the First Player

Intuitively, one would assume that the first player has an advantage over the second player because it has more choice. Hence one might think that the first player is always able to convince more nodes than the second player if it selects its seed nodes carefully. Theorem 7 proves the contrary.

Theorem 7 *In a two player rumor game where both player select one node to initiate their rumor in the graph, the first player does not always win.*

Proof. We consider an instance of the rumor game where both the first and the second player can select one node each as a seed. See Figure 4 for an example where the second player can always persuade more players than the first player regardless of the decision the first player makes. If player 1 chooses the node x_0 in the middle, the second player can select x_1 thus ensuring that 7 nodes believe rumor 2 and only 5 nodes adopt rumor 1. If player 1 decides for node x_1, player 2 can outwit the first player by choosing x_2. If player 1 designates x_2 as its seed, the second player select z_1. All other strategies are symmetric to one of the options mentioned or even less promising for player 1. Hence the second player can always convince 7 nodes whereas the first player has to content itself with 5 persuaded nodes. □

Since there exists no Condorcet Node in the graph in Figure 4, the curious reader might wonder whether a Condorcet Node guarantees at least $n/2$ convinced nodes for the first player. This conjecture is also wrong as Figure 5 demonstrates.

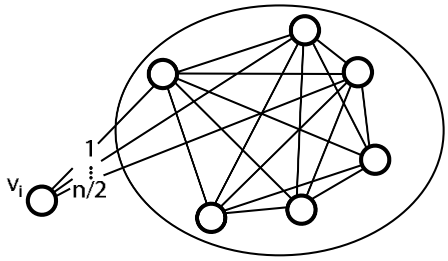

Fig. 5. Example where a Condorcet Node v_i yields a low payoff for player 1. The subgraph in the circle is a complete graph.

4.4 Heuristics for Centroid

Having discovered that there are graphs were the second player always is more successful in distributing its rumor, we now concentrate on games where the first player could convince more nodes than the second player. Since determining a centroid is NP-complete we consider the following (efficiently computable) strategies the first player can pursue: choose the node with smallest radius, with largest degree or the midpoint of the minimal spanning tree. The node with minimal maximal distance to any other node is the midpoint of the spanning tree. However, for these strategies Figure 3 shows examples where they do not win. In the example shown in Figure 6(a) player 1 selects the node v_j with the smallest radius rad_{min}, i.e., the minimum over all nodes v of the greatest distance between v and any other node. In this case the second player wins more than player 1 by choosing the highest degree node v_i, if it holds $degree(v_i) > 3 \cdot rad_{min}/2$. In Figure 6(b) player 1 selects the node v_i with highest degree. If it holds $n > 2 \cdot degree(v_i)$ then player 2 wins more than half of the nodes by selecting the neighbor of v_i. When the midpoint of a spanning tree is chosen by player 1 then it is easy to see that player 2 can choose a neighbor and win more

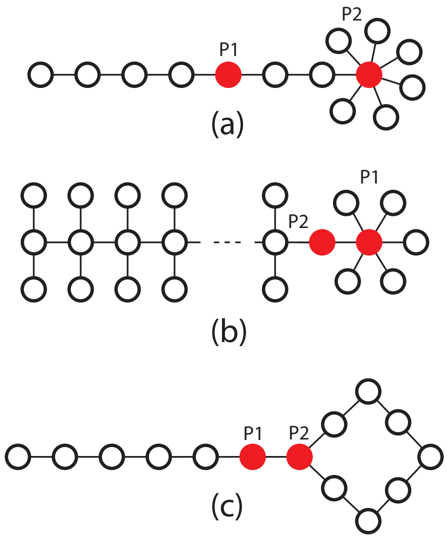

Fig. 6. Counterexamples for heuristics where player 1 wins fewer nodes than player 2 (a) Player 1 selects the node with smallest radius. (b) Player 1 selects the node with highest degree. (c) Player 1 selects the midpoint of the minimum spanning tree.

than half of the nodes, compare Figure 6(c). For all these heuristics there even exist graphs where the first player wins three nodes and the remaining nodes adopt the second rumor.

5 Conclusion

In this paper we have presented the rumor game which models the dissemination of competing information in networks. We defined a model for the game and specified how the propagation of the rumors in the network takes place. We proved that even for a restricted model computing the $(r|p)$-medianoid and $(r|p)$-centroid and its approximation is NP-complete. Moreover, we demonstrated the weaknesses of some heuristics for finding the centroid. Finally we proved the surprising fact that the first player does not always win our two-player rumor game, even when applying optimal strategies.

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