

On Objective Conflicts and Objective Reduction in Multiobjective Optimization

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Aachen, October 2, 2006

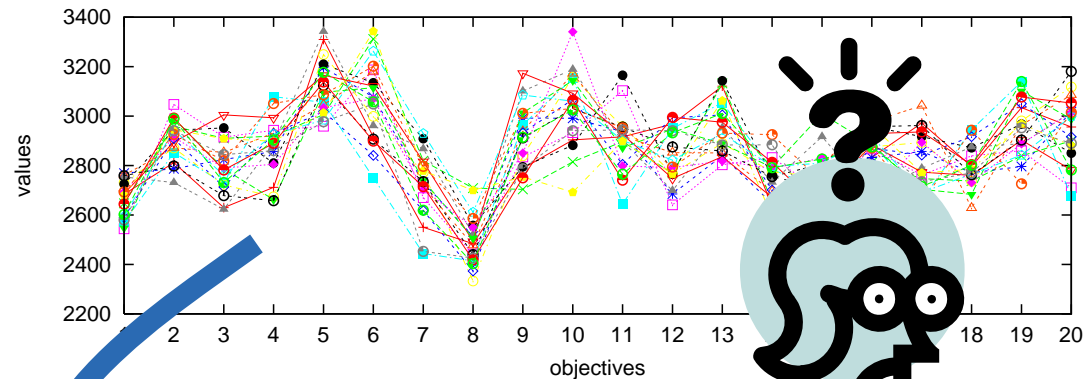
Motivation

Multiobjective Problem

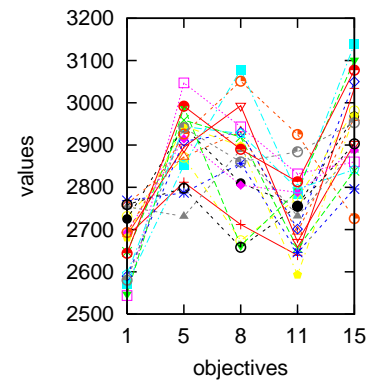
$\min. \{f_1(\mathbf{x}), \dots, f_k(\mathbf{x})\}$
subject to $\mathbf{x} \in S$
and $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$

Generating method

(Approximation of) efficient set

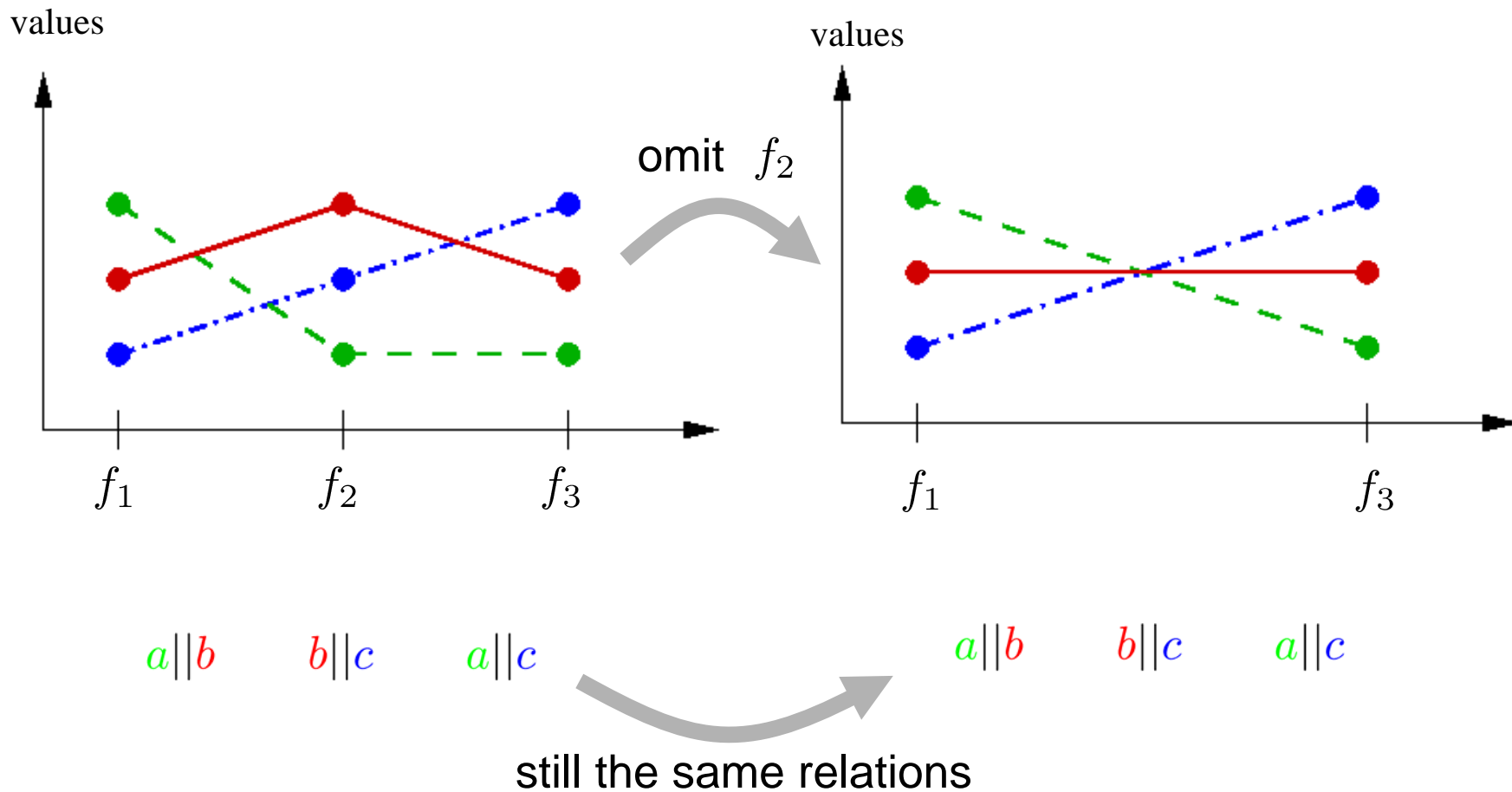


reduce number of objectives



⇒ assist decision maker

Example



- Objective Reduction in Decision Making Step
 - Objective reduction possible without changing/slightly changing the problem?
 - How to compute a minimum objective set?
- Objective Reduction During Search
 - How can a objective reduction method be used within the search?
 - Is objective reduction suitable in general?
 - What's the problem structure "on the way towards the Pareto front"?

- Objective Reduction in Decision Making Step
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- Objective reduction possible without changing/slightly changing the problem?
 - How to describe conflicts between objective sets?
- How to compute a minimum objective set?
 - Can we guarantee a lower bound on the error we make?

- **Omitting redundant objectives:**
 - Agrell (1997), Gal and Leberling (1977)
 - Not suitable for black-box optimization
- **PCA based objective reduction:**
 - Deb and Saxena (2005)
 - Cannot guarantee preservation of dominance structure
- **Various conflict definitions:**
 - Deb (2001); Tan et al. (2005)
 - conflict as a property of the problem itself
 - Purshouse and Fleming (2003):
 - *objective pairs* conflict if ≥ 2 solutions incomparable wrt the objective pair

Open Questions

- Conflicts between arbitrary objective sets
- Objective reduction with
 - preservation of problem structure
 - slight changes in problem structurein a black-box scenario
- “Real“ problems

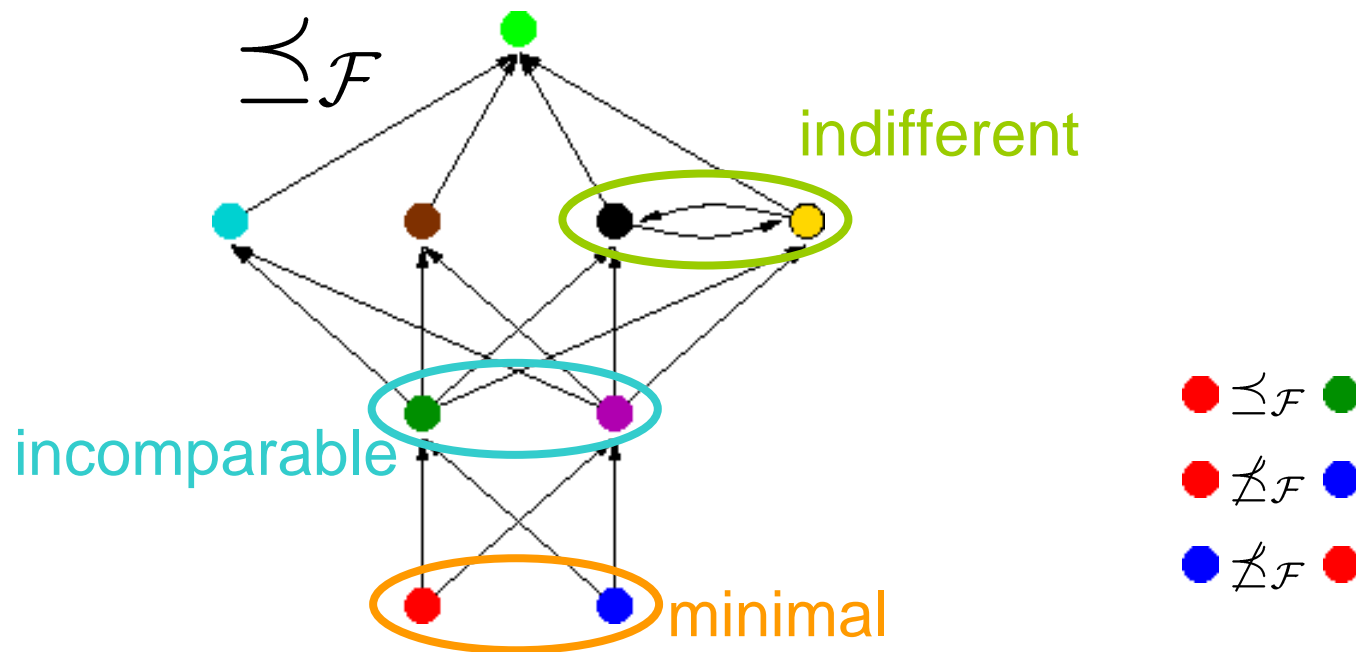
- Generalization of Objective Conflicts
- The Minimum Objective Subset Problems
 - Exact and heuristic algorithms
- Objective reduction for selected problems

- **Generalization of Objective Conflicts**
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Relation Graphs and Dominance

- For a multiobjective problem, the question is to find the minimal elements of a given (pre)order (X, \leq)
- Here, we restrict to the weak dominance relation $\preceq_{\mathcal{F}}$

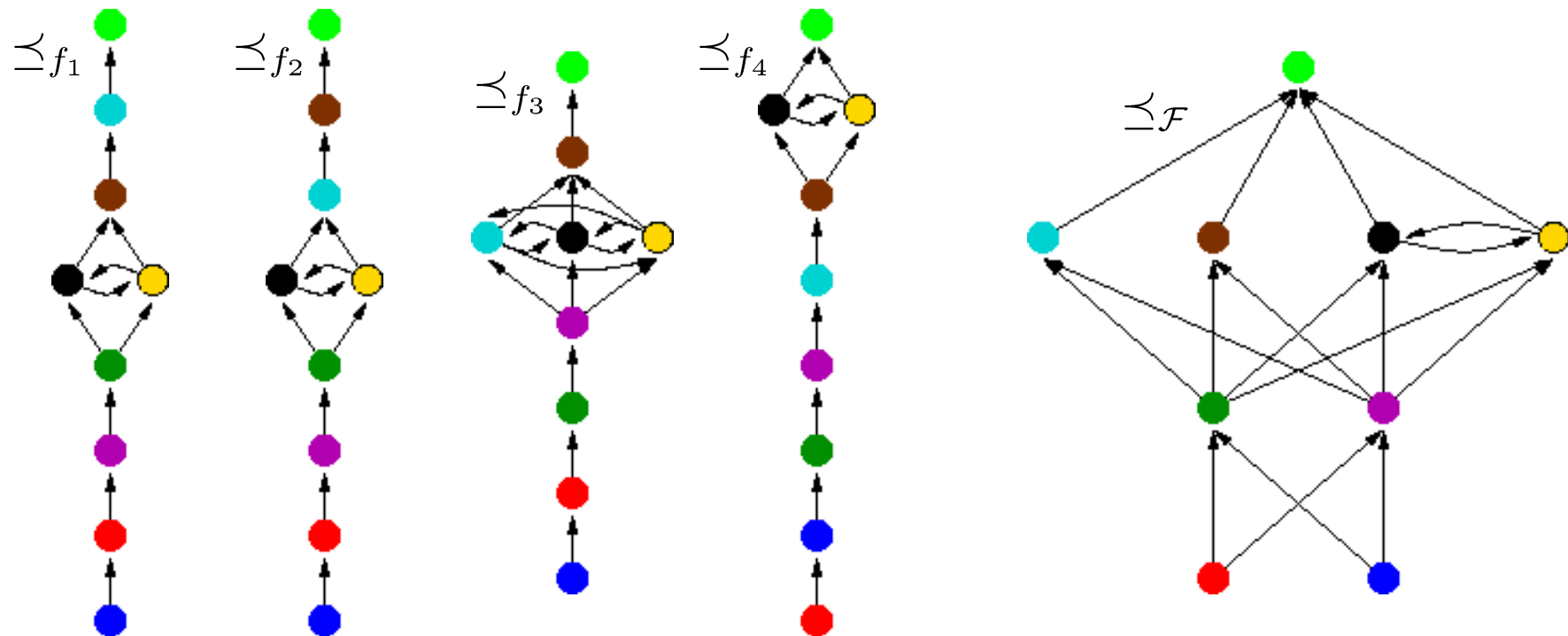
$$\preceq_{\mathcal{F}} := \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in X \wedge \forall f_i \in \mathcal{F} : f_i(\mathbf{x}) \leq f_i(\mathbf{y})\}$$



(reflexive and transitive edges are omitted)

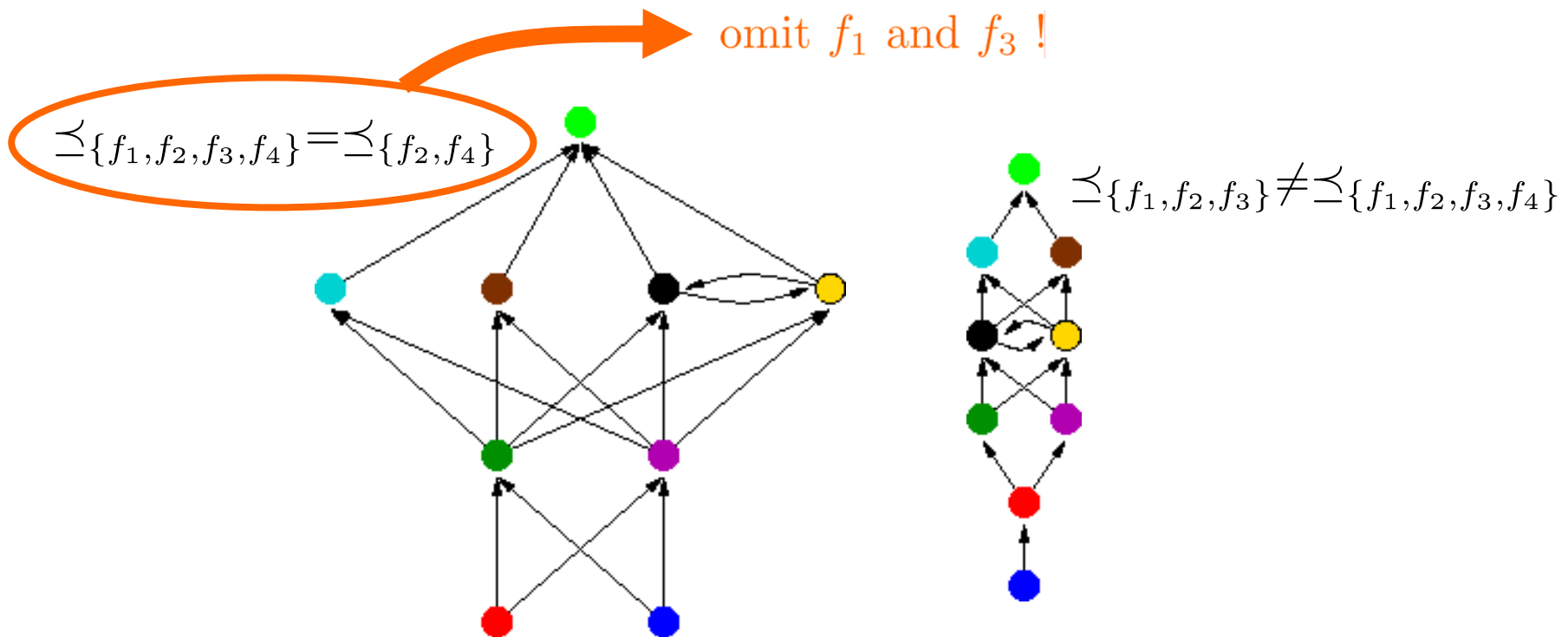
Intersection of Linear (Pre)Orders

- Single objectives induce linear (pre)orders \preceq_{f_i}
- Their intersection yields $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$
- Thus, the omission of objectives can only
 - make incomparable solution pairs comparable and
 - comparable solutions indifferent
 - add edges in relation graph



Objective Conflicts

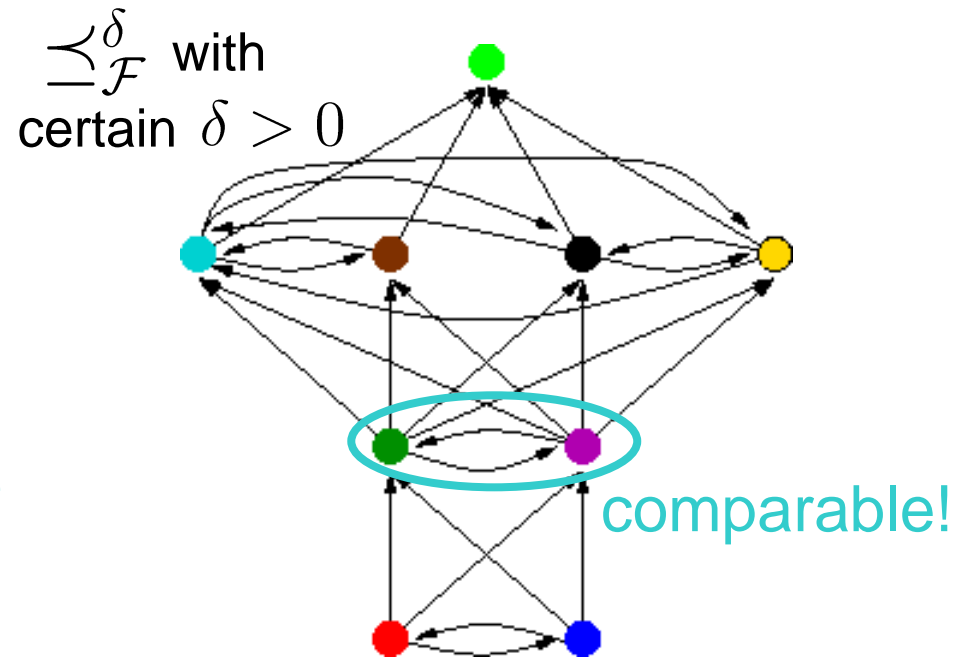
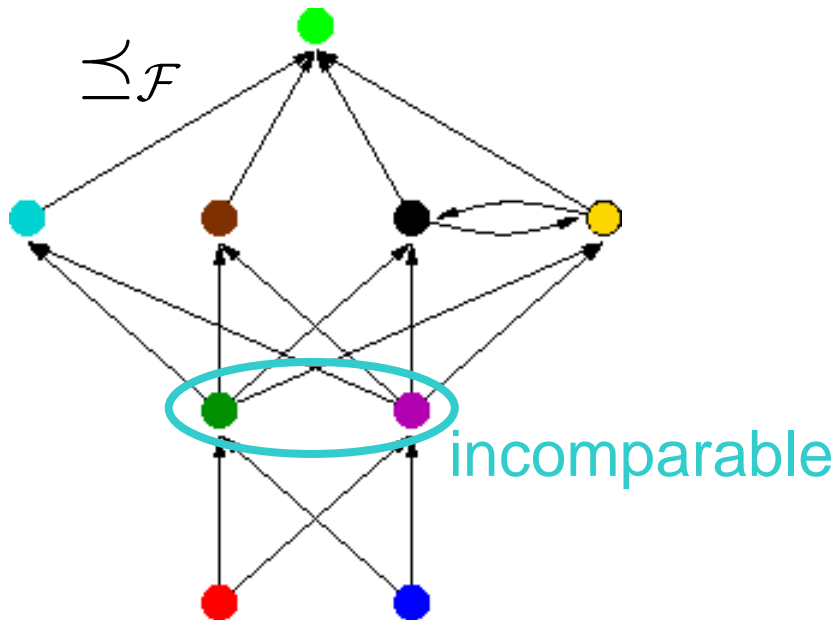
- Objective sets conflict if they induce different relations
 - Definition:** \mathcal{F}_1 nonconflicting with \mathcal{F}_2 iff $\preceq_{\mathcal{F}_1} = \preceq_{\mathcal{F}_2}$
 - Omit objectives in $\mathcal{F} \setminus \mathcal{F}'$ if $\mathcal{F}' \subseteq \mathcal{F}$ is nonconflicting with \mathcal{F} and preserve the dominance structure



Generalization of Objective Conflicts

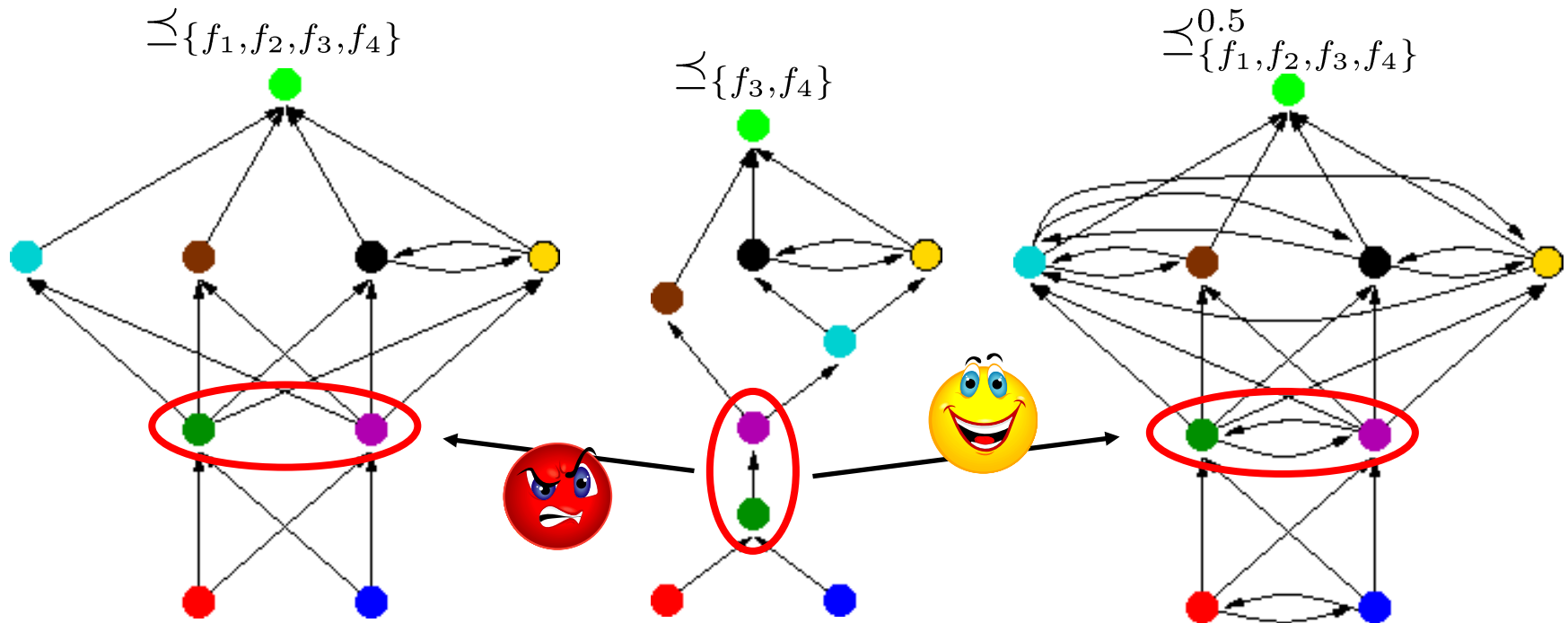
- Sometimes the limitation of preserving the problem structure is too strict
- Generalization to δ -conflict based on ε -dominance relation needed (now, objective values are used)

$$\preceq_{\mathcal{F}}^{\delta} := \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in X \wedge \forall f_i \in \mathcal{F} : f_i(\mathbf{x}) - \delta \leq f_i(\mathbf{y})\}$$



δ -Conflict

- Definition:** \mathcal{F}_1 δ -nonconflicting with \mathcal{F}_2 iff $\preceq_{\mathcal{F}_1} \subseteq \preceq_{\mathcal{F}_2}^{\delta}$ and $\preceq_{\mathcal{F}_2} \subseteq \preceq_{\mathcal{F}_1}^{\delta}$
- Omission of objectives in $\mathcal{F} \setminus \mathcal{F}'$ if $\mathcal{F}' \subseteq \mathcal{F}$ is δ -nonconflicting with \mathcal{F} guarantees that $\mathbf{x} \preceq_{\mathcal{F}}^{\delta} \mathbf{y}$ whenever $\mathbf{x} \preceq_{\mathcal{F}'} \mathbf{y}$



Key Contributions

- Generalization of Objective Conflicts
- **The Minimum Objective Subset Problems**
 - **Exact and heuristic algorithms**
- Objective reduction for selected problems

The Minimum Objective Subset Problem

Minimum objective set

$\mathcal{F}' \subseteq \mathcal{F}$ is called minimum if $\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}$

and $\nexists \mathcal{F}'' \subseteq \mathcal{F} \wedge |\mathcal{F}''| < |\mathcal{F}'| : \preceq_{\mathcal{F}''} = \preceq_{\mathcal{F}}$

Minimum Objective Subset Problem (MOSS)

Given: Set A of solutions with weak dominance relations

$$\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i} \quad \text{and} \quad \preceq_{f_i} \subseteq A \times A$$

Task: Compute a minimum objective set $\mathcal{F}' \subseteq \mathcal{F}$ with

$$\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}$$

MOSS is NP-hard

- Reduction from set cover problem (SCP)
- As a result, consideration of objective sets of fixed size is not sufficient

Generalized Minimum Objective Subset Problems

δ -Minimum objective set

$\mathcal{F}' \subseteq \mathcal{F}$ is called δ -minimum if $\preceq_{\mathcal{F}'} = \preceq_{\mathcal{F}}^{\delta}$, $\forall \delta' < \delta : \preceq_{\mathcal{F}'} \neq \preceq_{\mathcal{F}}^{\delta'}$
and $\nexists \mathcal{F}'' \subseteq \mathcal{F} \wedge |\mathcal{F}''| < |\mathcal{F}'| : \preceq_{\mathcal{F}''} = \preceq_{\mathcal{F}}^{\delta}$

δ -Minimum Objective Subset Problem (δ -MOSS)

Given: Set A of solutions with weak dominance relations
 $\preceq_{f_i} \subseteq A \times A$ and $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$ and a $\delta \geq 0$

Task: Compute a δ -minimum objective set $\mathcal{F}' \subseteq \mathcal{F}$ wrt \mathcal{F}

Objective Subset of size k with minimum error (kEMOSS)

Given: Set A of solutions with weak dominance relations
 $\preceq_{f_i} \subseteq A \times A$ and $\preceq_{\mathcal{F}} = \bigcap_{f_i \in \mathcal{F}} \preceq_{f_i}$ and a k

Task: Compute an objective subset $\mathcal{F}' \subseteq \mathcal{F}$, δ -nonconflicting
with \mathcal{F} , $|\mathcal{F}'| \leq k$ and minimal δ

Algorithms for the MOSS Problem

Exact algorithm

- Correctness proof
- Runtime: $O(|A|^2 \cdot k \cdot 2^k)$
- Worst case: $\Omega(|A|^2 \cdot 2^{k/3})$

```

1: Init:
2:    $M := \emptyset, S_M := \emptyset$ 
3:   for all pairs  $\mathbf{x}, \mathbf{y} \in A, \mathbf{x} \neq \mathbf{y}$  of solutions do
4:      $S_{\{\mathbf{x}, \mathbf{y}\}} := \emptyset$ 
5:     for all objective pairs  $i, j \in \mathcal{F}$ , not necessary  $i \neq j$  do
6:       compute  $\delta_{ij} := \delta_{\min}(\{i\} \cup \{j\}, \mathcal{F})$  wrt  $\mathbf{x}, \mathbf{y}$ 
7:        $S_{\{\mathbf{x}, \mathbf{y}\}} := S_{\{\mathbf{x}, \mathbf{y}\}} \sqcup (\{i\} \cup \{j\}, \delta_{ij})$ 
8:     end for
9:      $S_{M \cup \{\mathbf{x}, \mathbf{y}\}} := S_M \sqcup S_{\{\mathbf{x}, \mathbf{y}\}}$ 
10:     $M := M \cup \{\mathbf{x}, \mathbf{y}\}$ 
11:  end for
12: Output for  $\delta$ -MOSS:
13:    $(s_{\min}, \delta_{\min})$  in  $S_M$  with minimal size  $|s_{\min}|$  and  $\delta_{\min} \leq \delta$ 
14: Output for kEMOSS:
15:    $(s, \delta)$  in  $S_M$  with size  $|s| \leq k$  and minimal  $\delta$ 
  
```

Simple greedy heuristics

- Correctness proof
- Runtime
 - $O(\min\{k^3 \cdot |A|^2, k^2 \cdot |A|^4\})$ (δ MOSS)
 - $O(k^3 \cdot |A|^2)$ (kEMOSS)
- Best possible approximation ratio of $\Theta(\log |A|)$ for the case $\delta = 0$

```

1: Init:
2:   compute the relations  $\preceq_i$  for all  $1 \leq i \leq k$  and  $\preceq_{\mathcal{F}}$ 
3:    $\mathcal{F}' := \emptyset$ 
4:    $R := A \times A \setminus \preceq_{\mathcal{F}}$ 
5:   while  $R \neq \emptyset$  do
6:      $i^* = \operatorname{argmin}_{i \in \mathcal{F} \setminus \mathcal{F}'} \{ |(R \cap \preceq_i) \setminus (\preceq_{\mathcal{F}' \cup \{i\}}^0 \cap \preceq_{\mathcal{F}' \setminus \{i\}}^\delta)| \}$ 
7:      $R := (R \cap \preceq_{i^*}) \setminus (\preceq_{\mathcal{F}' \cup \{i^*\}}^0 \cap \preceq_{\mathcal{F}' \setminus \{i^*\}}^\delta)$ 
8:      $\mathcal{F}' := \mathcal{F}' \cup \{i^*\}$ 
9:   end while
  
```

δ -MOSS

```

1: Init:
2:    $\mathcal{F}' := \emptyset$ 
3:   while  $|\mathcal{F}'| < k$  do
4:      $\mathcal{F}' := \mathcal{F}' \cup \operatorname{argmin}_{i \in \mathcal{F} \setminus \mathcal{F}'} \{ \delta_{\min}(\mathcal{F}' \cup \{i\}, \mathcal{F}) \text{ wrt } A \}$ 
5:   end while
  
```

kEMOSS

Key Contributions

- Generalization of Objective Conflicts
- The Minimum Objective Subset Problems
 - Exact and heuristic algorithms
- **Objective reduction for selected problems**

Objective Reduction for Selected Problems

No error:

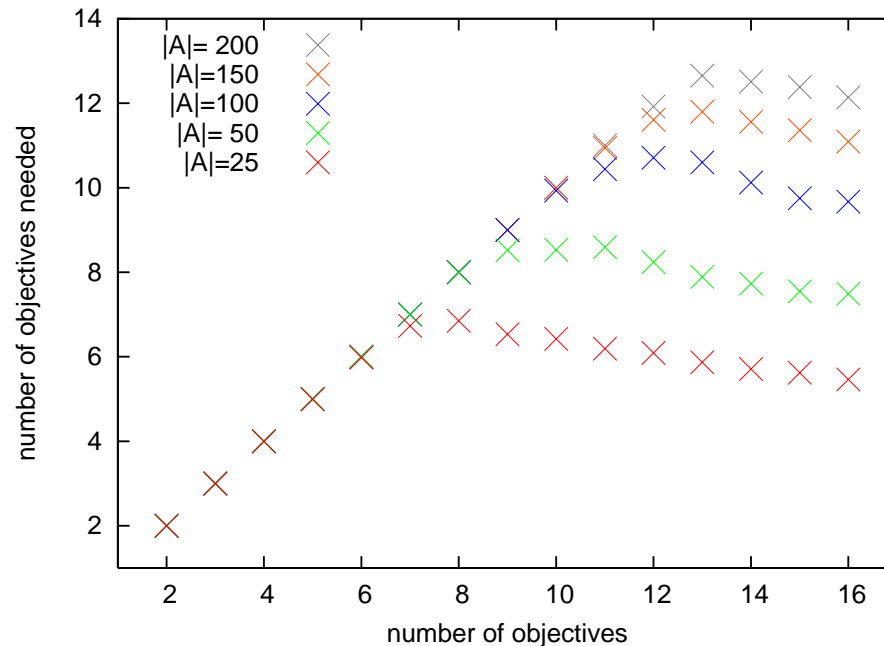
- Solutions with randomly chosen objective values (i.e., random orders as \preceq_{f_i}):
 - Objective reduction possible?
 - Size of minimum set influenced by solution set size and number of objective?
 - Greedy vs. exact algorithm
- Realistic scenarios for test problems

Influence of δ and k :

- Comparison between greedy and exact algorithms

Varying $|A|$ and k for Random Orders

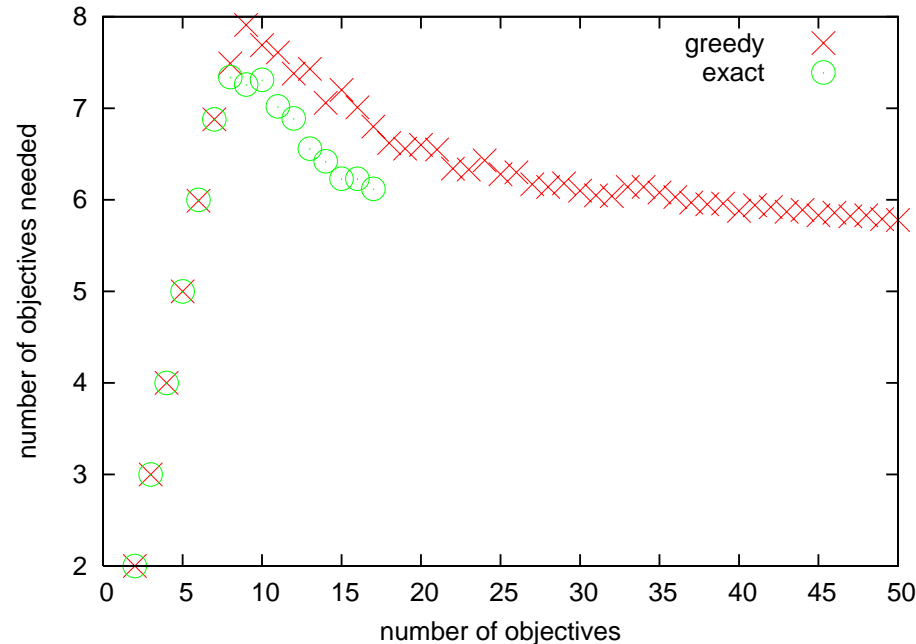
Various solution set sizes $|A|$ with random orders as \preceq_{f_i}



- The more objectives, the smaller the minimum sets
- The more solutions in A , the fewer objectives omissable

Greedy vs. Exact Algorithm for Random Orders

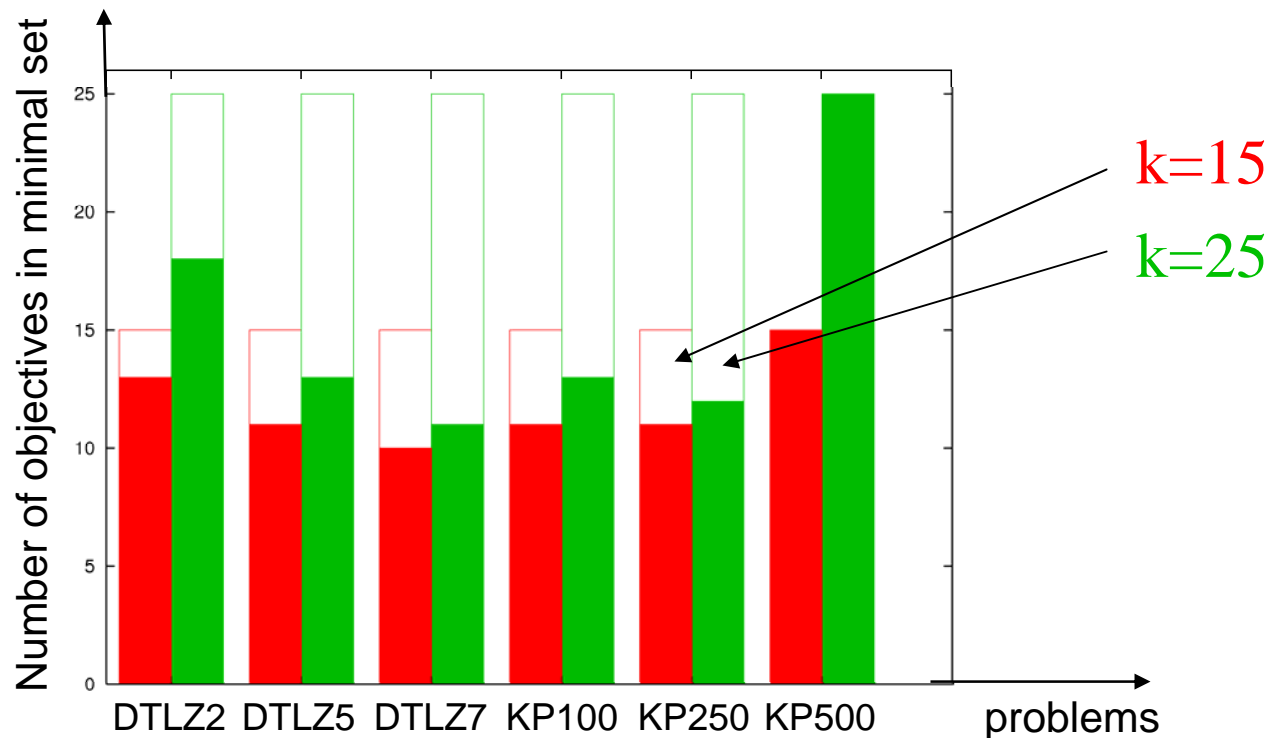
Heuristic vs. exact algorithm on random orders \preceq_{f_i} with $|A| = 32$



- The greedy algorithm's objective sets are not too large
- Greedy algorithm has clearly lower running time:
 - can handle 50 objectives instead of ≤ 20 compared to exact algorithm within the same time

Realistic Scenarios for Test Problems

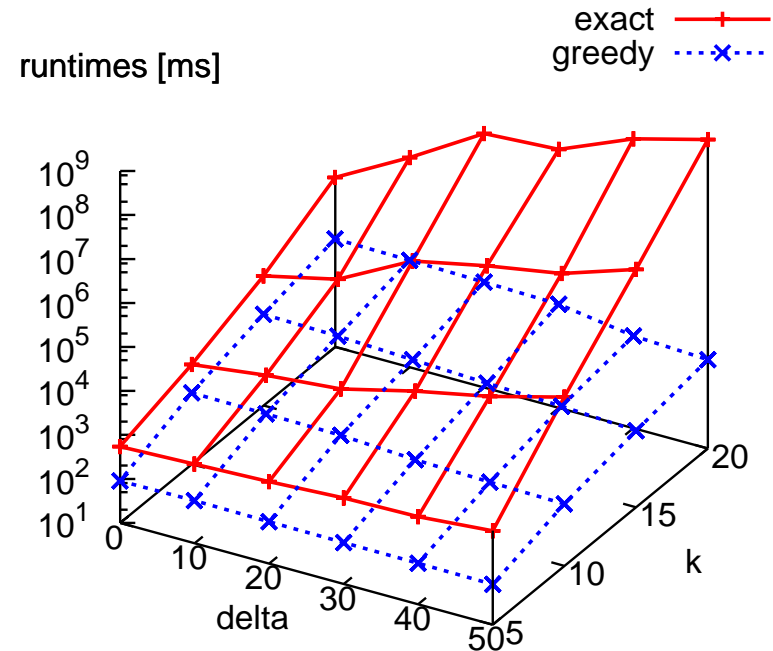
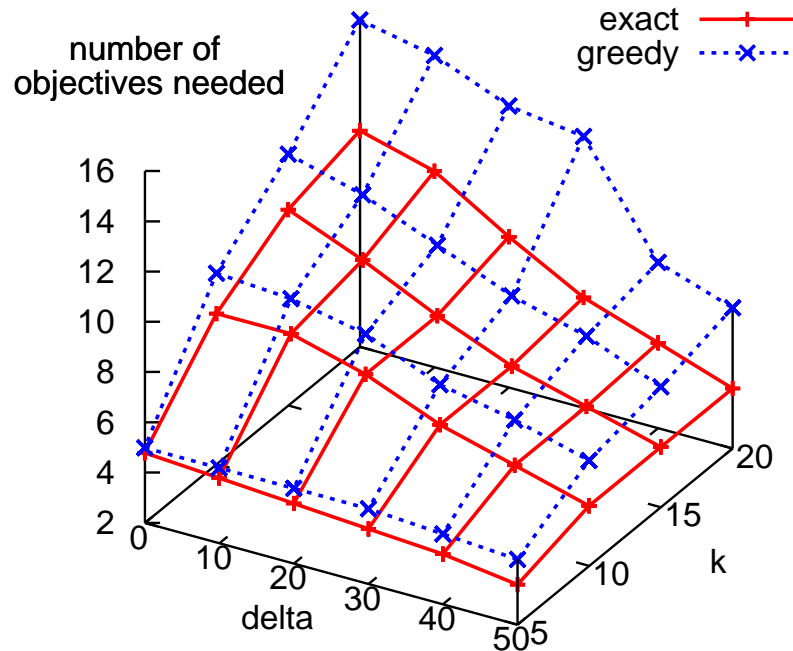
- Approximation of efficient set computed by evolutionary algorithm used as A
- $|A| = 200$ for $k = 15$ and $|A| = 300$ for $k = 25$



- Objective reduction of $\leq 50\%$ possible for various test problems

Comparison of Algorithms for δ -MOSS

Entire Search Space of 0-1-Knapsack Problem with 7 Items



- heuristic slightly worse results, but clearly faster
- ⇒
- the more objectives, the more objectives can be omitted
 - The larger the error, the smaller the sets

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Problems with Many Objectives

- MOEAs, working on 2D and 3D problems are not suitable for many objective optimization (NSGA-II, SPEA2, ...)
 - Why?
 - Not clear in general
 - Number of incomparable solution pairs increases
- Widely believed, that problems become harder with more objectives

Reducing number of objectives:

- Maneeratana et al. (2006):
 - Reducing of MOP to 2D-problem (drawback: new objectives, no preservation of dominance relation)
- Deb and Saxena (2005):
 - Multiple starts of NSGA-II with reduced number of objectives, choice of objectives based on PCA

General Investigations:

- Neumann and Wegener (2006), Scharnow et al. (2002):
 - Few examples where more objectives help
- But nearly every textbook says that more objectives makes the problem harder, e.g., Deb (2001)
- P. Winkler (1985):
 - Random orders as objectives with n points in k dimensions
 - Width between $e^{-1}n^{(k-1)/k}$ and $n^{(k-1)/k} \ln(n)$

Reducing Number of Objectives Within Search:

- How to include (adaptive) objective reduction into EA while using subset of given objectives?

General Investigations:

- Does all problems become harder with more objectives?
- Is it due to more incomparable solutions?

Reducing Number of Objectives Within Search

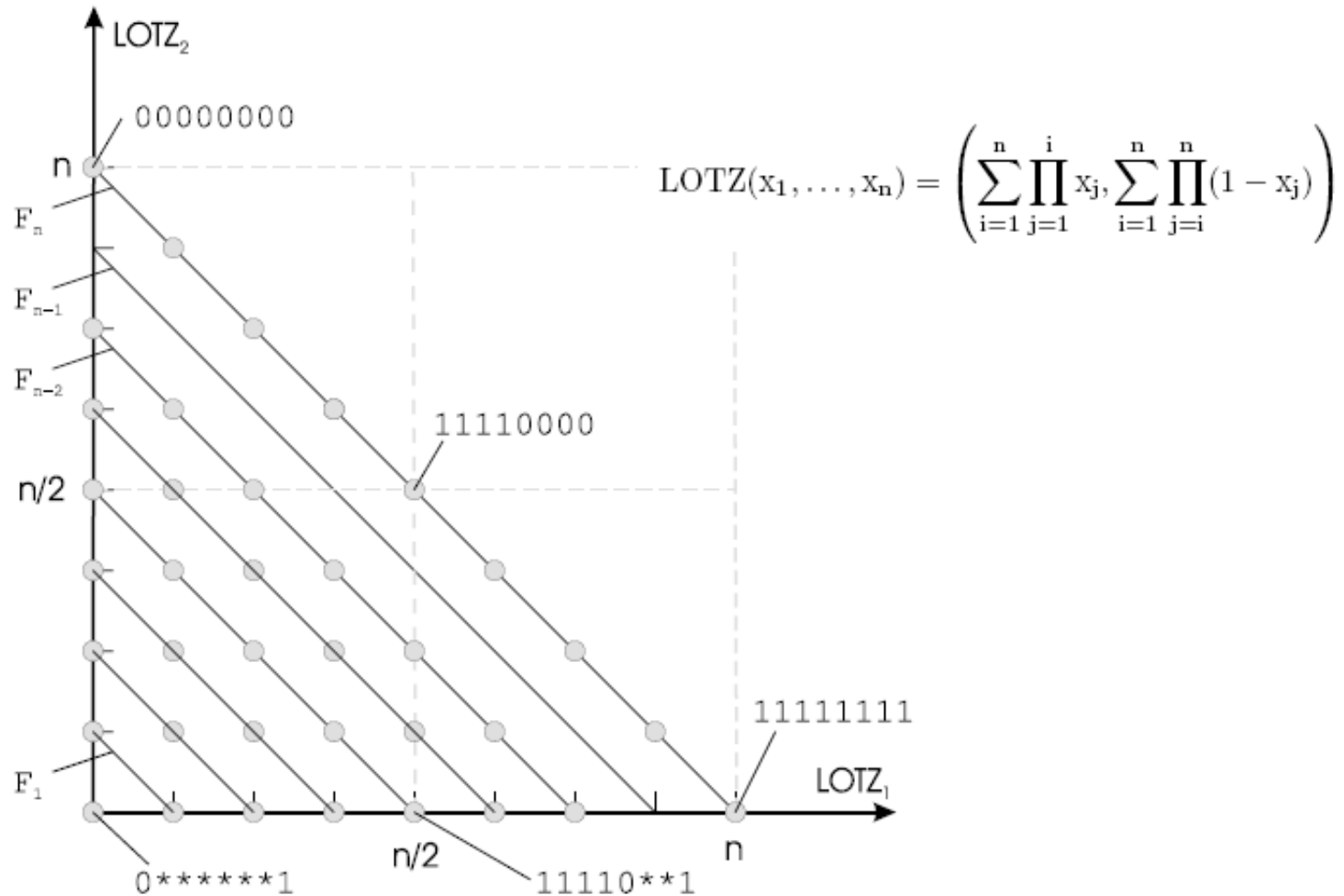
- If EA detects, that objectives can be omitted, then objective reduction is not necessary any more
 - Exception: objective function evaluations are expensive
- Problem is not the number of objectives but the number of incomparable solutions
 - No direction to better solutions observable
 - Potential way out:
 - use indicator to refine Pareto dominance relation (e.g. Hypervolume indicator/S-metric/Lebesgue-measure)

General Investigations

Do all problems become harder with more objectives?
Is it due to more incomparable solutions?

- 4 simple (toy) problems based on 2D problem
- LOTZ, resp. modified LOTZ
- Add third objective
 - This can both increase or decrease the difficulty of the problem
 - Both when
 - making indifferent solutions comparable, and
 - making comparable solutions incomparable!

LOTZ - Leading Ones Trailing Zeros



Third Objectives Makes Indifferent Comparable

- Problem 1 (harder than LOTZ):

$$f_1(\mathbf{x}) := \text{LEADING ONES}(\mathbf{x})$$

$$f_2(\mathbf{x}) := \text{TRAILING ZEROS}(\mathbf{x})$$

$$f_3(\mathbf{x}) := |\mathbf{x}_M| - \text{LEADING ONES}(\mathbf{x}_M) - \text{TRAILING ZEROS}(\mathbf{x}_M)$$

- Problem 2 (easier than LOTZ):

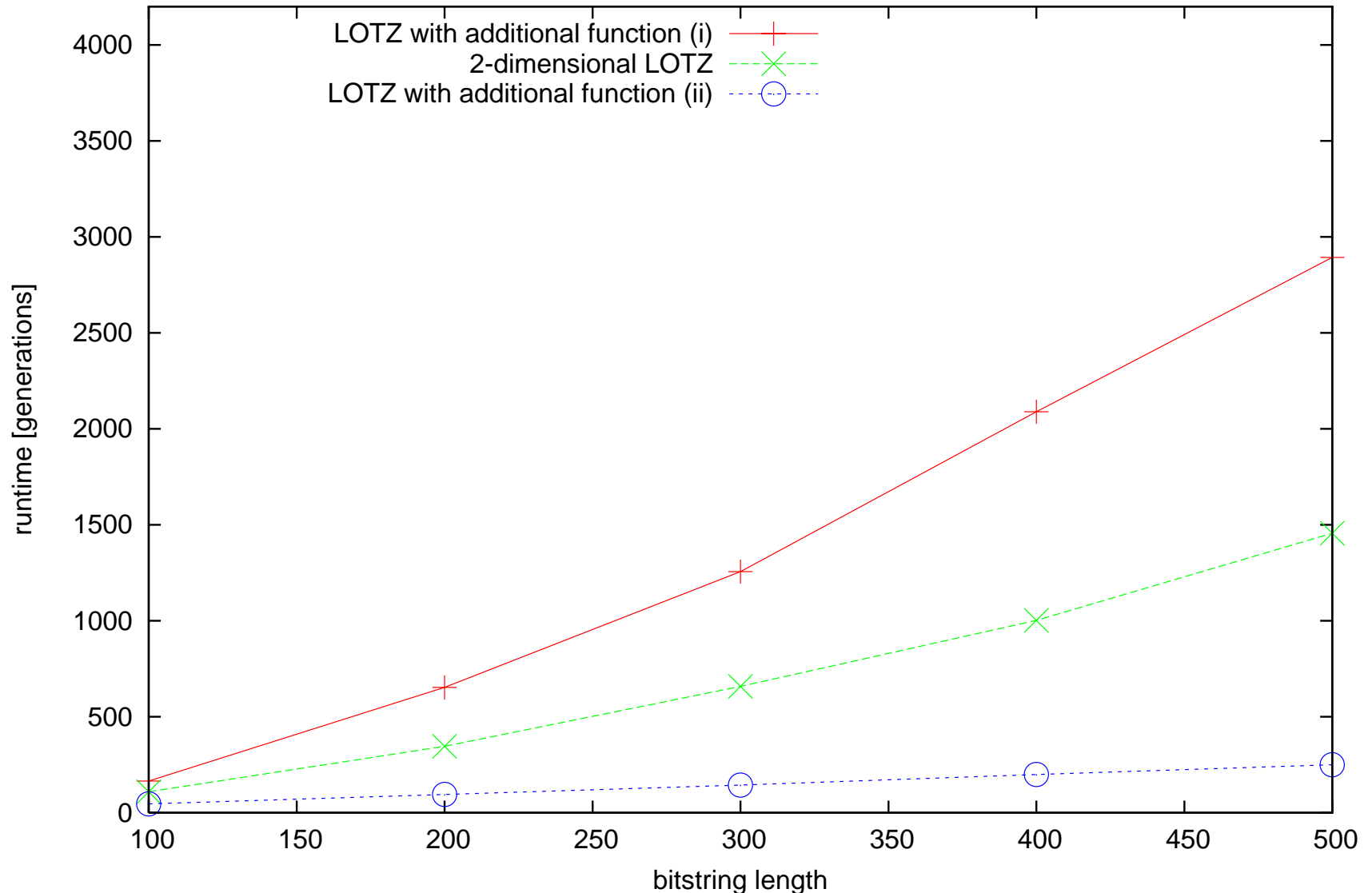
$$f_1(\mathbf{x}) := \text{LEADING ONES}(\mathbf{x})$$

$$f_2(\mathbf{x}) := \text{TRAILING ZEROS}(\mathbf{x})$$

$$f_3(\mathbf{x}) := \text{ONEMAX}(\mathbf{x}_M)$$

Third Objectives Makes Indifferent Comparable (2)

Average runtimes for 10 IBEA runs with population size 200



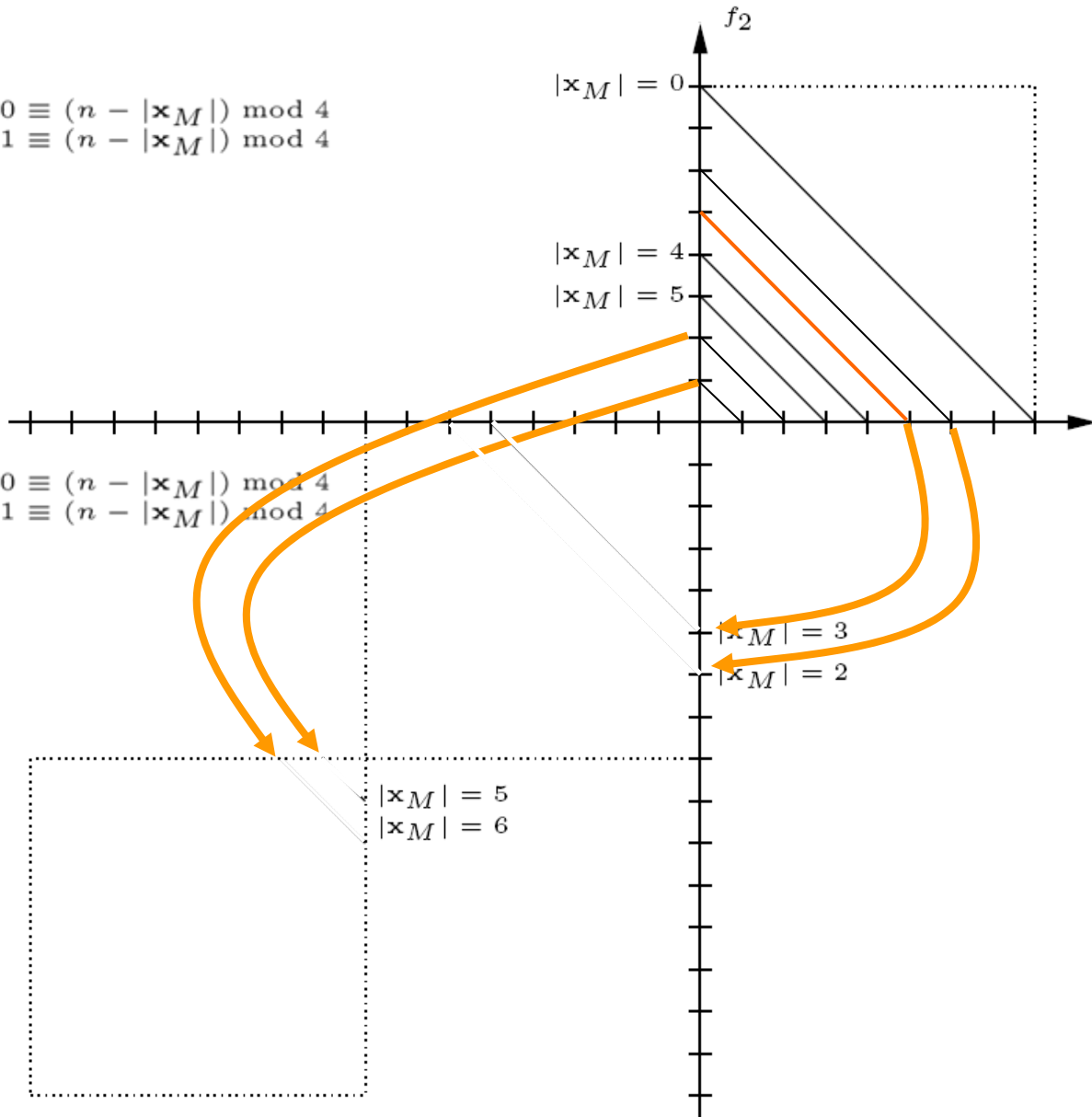
Modified LOTZ

moL0(\mathbf{x}) =

$$\begin{aligned} & L0(\mathbf{x}) && \text{iff} && 0 \equiv (n - |\mathbf{x}_M|) \pmod{4} \\ & \text{or} && && 1 \equiv (n - |\mathbf{x}_M|) \pmod{4} \\ (-n) \cdot \frac{n - |\mathbf{x}_M|}{2} - L0(\mathbf{x}) && \text{else} \end{aligned}$$

moTZ(\mathbf{x}) =

$$\begin{aligned} & TZ(\mathbf{x}) && \text{iff} && 0 \equiv (n - |\mathbf{x}_M|) \pmod{4} \\ & \text{or} && && 1 \equiv (n - |\mathbf{x}_M|) \pmod{4} \\ (-n) \cdot \frac{n - |\mathbf{x}_M|}{2} - TZ(\mathbf{x}) && \text{else} \end{aligned}$$



Third Objectives Makes Comparable Incomparable

- Problem 3 (harder than modified LOTZ):

$$f_1(\mathbf{x}) := \text{modified LEADING ONES}(\mathbf{x})$$

$$f_2(\mathbf{x}) := \text{modified TRAILING ZEROS}(\mathbf{x})$$

$$f_3(\mathbf{x}) := \frac{n}{2} - \left| \frac{n}{2} - \|\mathbf{x}_M\| \right|$$

- Problem 4 (easier than modified LOTZ):

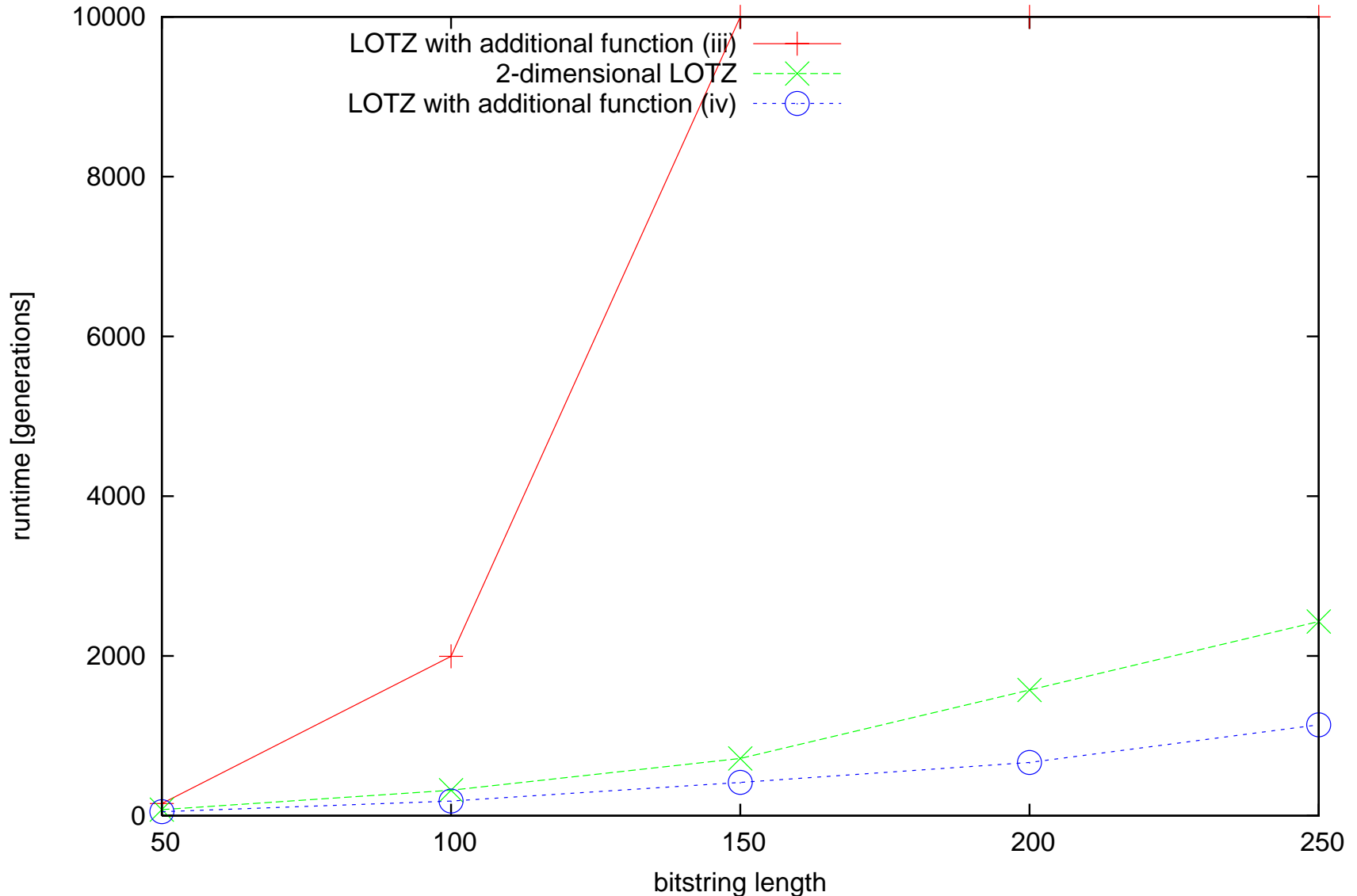
$$f_1(\mathbf{x}) := \text{modified LEADING ONES}(\mathbf{x})$$

$$f_2(\mathbf{x}) := \text{modified TRAILING ZEROS}(\mathbf{x})$$

$$f_3(\mathbf{x}) := \|\mathbf{x}_M\|$$

Third Objectives Makes Comparable Incomparable (2)

Average runtimes for 10 IBEA runs with population size 100

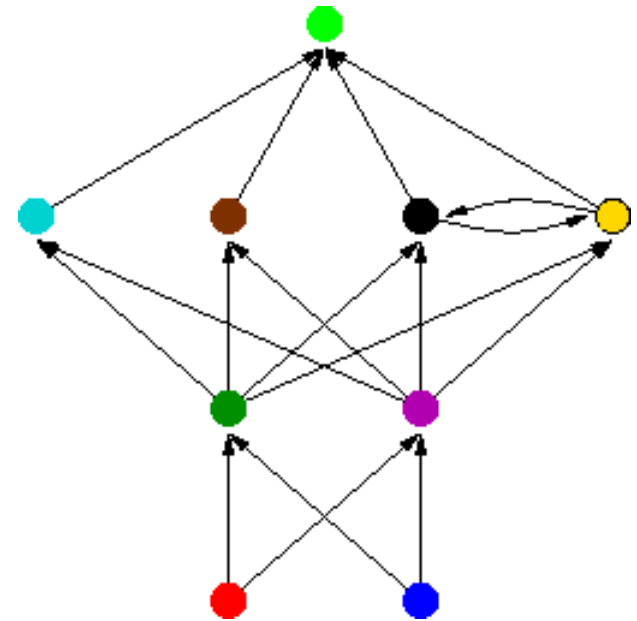
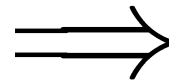
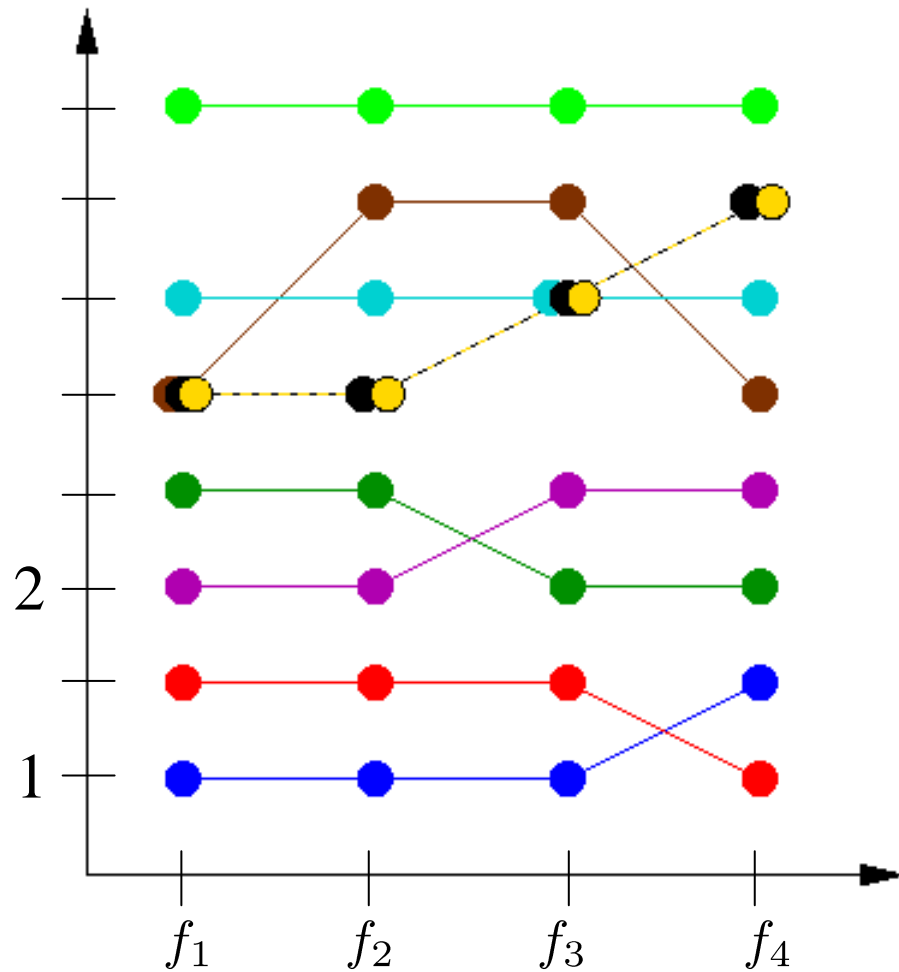


Conclusions and Outlook

- Generalization of Objective Conflicts
- The MOSS Problem and algorithms
- Method feasible for decision making process for selected problems
 - Also for real world problems?
- General discussion of problems with many objectives
 - Current work: general indicator properties

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Parallel Coordinates Plot for Example



Parallel Coordinates Plot for Example

