A Tight Lower Bound for the Capture Time of the Cops and Robbers Game

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Abstract

For the game of *Cops and Robbers*, it is known that in 1-cop-win graphs, the cop can capture the robber in $O(n)$ time, and that there exist graphs in which this capture time is tight. When $k \geq 2$, a simple counting argument shows that in k-cop-win graphs, the capture time is at most $O(n^{k+1})$, however, no non-trivial lower bounds were previously known; indeed, in their 2011 book, Bonato and Nowakowski ask whether this upper bound can be improved. In this paper, the question of Bonato and Nowakowski is answered on the negative, proving that the $O(n^{k+1})$ bound is asymptotically tight for any constant $k \geq 2$. This yields a surprising gap in the capture time complexities between the 1-cop and the 2-cop cases.

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1 Introduction

The game of *Cops and Robbers* is a perfect information two-player zero-sum game played on an undirected n-vertex graph $G = (V, E)$, where the first player is identified with $k \geq 1$ *cops*, indexed by the integers $0, \dots, k-1$, and the second player is identified with a single *robber*. In round 0, the cop player chooses the initial (not necessarily distinct) cop locations $c_0(0), \dots, c_{k-1}(0) \in V$ and following that, the robber player chooses the initial robber location $r(0) \in V$. Then, in round $t = 1, 2, \dots$, the cop player chooses the next (not necessarily distinct) cop locations $c_0(t), \dots, c_{k-1}(t) \in V$ under the constraint that $c_i(t) \in N^+(c_i(t-1))$ for every $0 \leq i \leq k-1$, where $N^+(v)$ denotes the neighborhood of vertex v in G including v itself; following that, the robber player chooses the next robber location $r(t) \in N^+(r(t-1))$.

The goal of the cop player is to ensure that $r(t-1) \in \{c_0(t), \dots, c_{k-1}(t)\}$ for some finite round t, referred to as *capturing* the robber. Conversely, the goal of the robber player is to avoid being captured indefinitely. Graph G is said to be a *k-cop-win* graph if it admits a cop strategy $\mathcal{S}$ that guarantees capture. The *capture time* of $\mathcal{S}$ is defined to be the maximum number of rounds until capture is achieved, assuming optimal play by the robber. The *capture time* of graph G is then defined to be the minimum capture time of any cop strategy over G (notice that in this definition, it is assumed that k is clear from the context).

Bonato et al. [7] studied the capture time in single cop games and proved that every 1-cop-win graph admits a cop strategy that captures the robber in $\mathcal{O}(n)$ rounds. By considering

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a path, it is straightforward to verify that this bound is asymptotically tight. A simple configuration-counting argument (see, e.g., [5, 7]) shows that for any constant $k \geq 2$, if G is a k-cop-win graph, then its capture time is $\mathcal{O}(n^{k+1})$. One may suspect that this simple upper bound can be improved as it does not generalize the tight $\mathcal{O}(n)$ bound in the 1-cop setting. Answering an open question from Bonato and Nowakowski's book [9, Chapter 8], the main result of our paper is that perhaps surprisingly, this is not the case.

▶ **Theorem 1.** *There exist a universal positive constant α such that for every $k \geq 2$, there exists an infinite family $\mathcal{G}$ of k-cop-win graphs such that the capture time of any n-vertex graph $G \in \mathcal{G}$ is at least $(n/(\alpha k))^{k+1}$. Moreover, the smallest graph in $\mathcal{G}$ has $n = \mathcal{O}(k^2)$ vertices.*

Notice that for constant $k \geq 2$, this theorem provides an (existential) $\Omega(n^{k+1})$ lower bound on the capture time in k-cop-win graphs. Furthermore, it can be extended to non-constant values of $k = k(n)$ up to the conjectured maximum of $k(n) = \Theta(\sqrt{n})$ (see the related literature discussion), stating that in some k-cop-win graphs, the capture time is exponential in k and stretched exponential in n.

2 Related Literature.

The Cops and Robbers game with a single cop was introduced by Quilliot [21] and independently by Nowakowski and Winkler [19] who also provided a full characterization of 1-cop-win graphs. This was generalized to the multiple cop setting by Aigner and Fromme [2] who defined the *cop number* of graph G to be the minimum number of cops that guarantees that the robber can be captured (that is, the minimum k for which G is a k-cop-win graph). Cast in this terminology, Aigner and Fromme proved that the cop number of any planar graph is at most 3. An upper bound of $\mathcal{O}(r^2)$ on the cop number of graphs excluding K_r as minor was established by Andreae [4]; this result lies at the heart of the recent graph decomposition technique of Abraham et al. [1] for the same family of graphs. For general graphs, the maximum possible cop number is still an open question: the famous Meyniel's Conjecture [14, 6] states that this number is $\Theta(\sqrt{n})$, where the state of the art is that it is bounded between $\Omega(\sqrt{n})$ [20] and $\mathcal{O}(n/2^{(1-o(1))\sqrt{\log n}})$ [17]. Several characterizations of graphs with cop number k are presented in [11].

As mentioned earlier, Bonato et al. [7] established a tight linear bound on the capture time in 1-cop-win graphs. For $k > 1$ cops, non-trivial bounds on the capture time in k-cop-win graphs were obtained mainly in the context of special graph classes, e.g., hypercubes [8] and Cartesian products of trees [18]. To the best of our knowledge, the linear lower bound of [7] is the (asymptotically) best previously known lower bound on the capture time in any class of graphs for any $k \geq 1$.

The capture time has been studied also for variants of the classic Cops and Robbers game. For example, the multiple robber setting was investigated by Förster et al. [13] who showed that the capture time may increase linearly with the number of robbers. Kehagias and Pralat [16] analyzed the expected capture time of a *drunk robber* whose strategy is simply a random walk on the graph. For a broader overview of the results on the game of Cops and Robbers, the reader is referred to the book of Bonato and Nowakowski [9] and recent surveys [3, 10, 12, 15].

Techniques.

Our lower bound proof relies on designing a bad (from the perspective of the cops) graph G that consists of several components, of which each has a different role (see the overview in Section 3.1). Here, we provide a glimpse into this design from an alternative (strictly informal and somewhat inaccurate) angle that may shed additional light on the techniques we use. At the heart of our construction lies the concept of forcing each entity (cop or robber) to follow a designated (non-simple) path in G, where in the scope of this discussion, we assume that every vertex in G admits a self-loop so that these paths may include null moves. Specifically, graph G contains equally long paths $\chi_0, \ldots, \chi_{k-1}$ and ρ, referred to in this discussion as the *canonical paths* of the cops and robber, respectively. The best strategy of the cop player is then to assign one cop, say Cop i, to each path χ_i so that $c_i(t) = \chi_i(t)$ for every t; in response to that, the best strategy of the robber is to play $r(t) = \rho(t)$. This induces a sequence σ of (distinct) configurations and the analysis is completed by showing that σ is sufficiently long and that the robber is captured only at its end.

The most challenging part in the design of such canonical paths is to prove that the aforementioned strategies are indeed optimal. To that end, we show that if the robber deviates from her canonical path at time t, then she is either captured immediately or the game shifts forward to a more advanced configuration $\sigma(t')$ for some $t' > t$. Conversely, if some cop deviates from her canonical path at time t, then the game shifts backwards to a less advanced configuration $\sigma(t')$ for some $t' < t$. The main feature in the latter argument is an *exit component* $\mathcal{X}$; if the robber manages to reach $\mathcal{X}$, then she can force the game to shift backwards to the beginning, i.e., to $\sigma(0)$. This threat is the key ingredient in the analysis of the cop strategy: we construct the cops' components so that they must strictly follow their canonical paths in order to cover all exits in $\mathcal{X}$.

Technical Remarks.

We call the set $\{c_0(t), c_1(t), \ldots, c_{k-1}(t)\}$, for some t, a *cop combination*. We say that Cop i *covers* node v at time t if $v \in N^+(c_i(t))$. We may omit t if it is clear from the context. We extend this covering notion to more than one cop and more than one node, e.g., we say that *the* cops cover a set of nodes $\{v_1, v_2, \ldots, v_j\}$ if each of the v_i is covered by at least one of the cops. Also, we say that a node is covered, resp. *uncovered*, if at least one cop covers it, resp. if no cop covers it.

Due to the verbosity of our construction and to the space limitations, we defer all the proofs to Appendix A and the discussion of the case of more than 2 cops to Appendix B.

3 The Case of 2 Cops

3.1 Overview

In this section, we construct a family $\{G_{\hat{n}}\}_{\hat{n} \geq 3, \hat{n} \equiv 0 \pmod 3}$ of 2-cop-win graphs, where $\hat{n} \in \Theta(n)$.[1] Then, we show that the capture time for 2 cops in $G_{\hat{n}}$ is $\Omega(n^3)$ by giving an explicit strategy for the robber that achieves this capture time against any cop strategy. We conclude by presenting a cop strategy for 2 cops that achieves a capture time of $\mathcal{O}(n^3)$ against any robber strategy, which also serves as a proof that the graphs are indeed 2-cop-win. The other reason for explicitly specifying such a cop strategy (which *must* exist in 2-cop-win

[1] Throughout the paper, we denote the number of nodes of a graph by n.

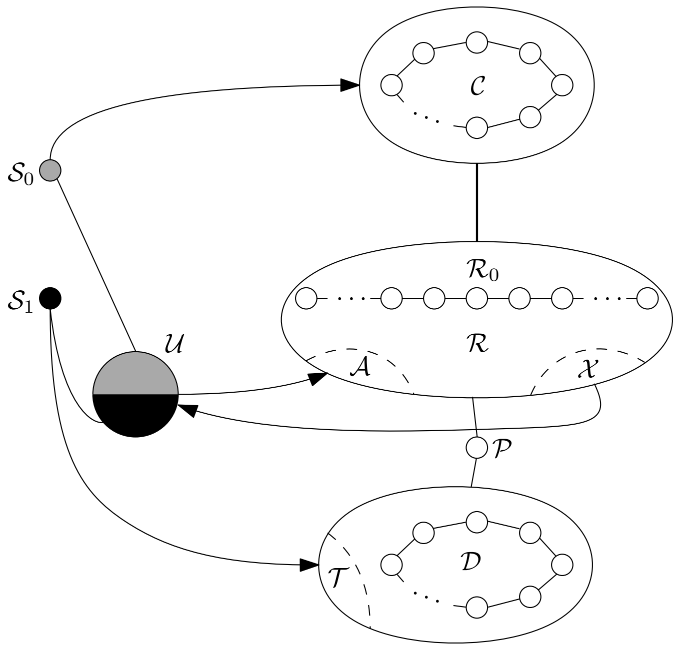

Figure 1 The graph $G_{\hat{n}}$ with its different components.

graphs, by the aforementioned simple configuration-counting argument) is that it forms the basis for a generalized cop strategy in the case of $k \geq 2$ cops. For a simplified overview of our graph construction, please refer to Figure 1.

The general idea behind the graph construction and the specified strategies for the cops and the robber is as follows: Our graph $G_{\hat{n}}$ contains a $\mathcal{U}$ component, where the robber cannot be captured, simply because each node in $\mathcal{U}$ has enough neighbors so that 2 cops cannot cover all of these neighbors simultaneously. Moreover, the robber can always stay in $\mathcal{U}$ (because each node in $\mathcal{U}$ has enough neighbors *in* $\mathcal{U}$) except if the cops go to two special nodes $\mathcal{S}_0$ and $\mathcal{S}_1$ which together cover all of $\mathcal{U}$.

When the robber is thereby flushed out of $\mathcal{U}$, she has to go to the $\mathcal{R}$ component of the graph. Note that the nodes of $\mathcal{R}$ induce a simple path on $G_{\hat{n}}$ and that after being flushed out of $\mathcal{U}$, the robber is located in the middle of this path. Now, the cops will continuously prevent the robber from escaping $\mathcal{R}$ and slowly force the robber to one end of the path where they will finally capture her. In order for this to take a long time, each node in $\mathcal{R}$ is connected to a set of so-called *exits* which are nodes that together form the $\mathcal{X}$ component of the graph. If the robber should manage to get to some exit, then she will be able to go back to her preferred $\mathcal{U}$ component, unless the cops go again to the special $\mathcal{S}_0$ and $\mathcal{S}_1$ nodes, in which case the robber can go back to the middle of the $\mathcal{R}$ path and thereby revert to a previous configuration. Hence, in order to capture the robber, the cops have to continuously cover these exits.

Unfortunately for the cops, there are only a few cop combinations that actually cover all exits of a node in $\mathcal{R}$. Moreover, only some of these cop combinations are *proper* in the sense that they also prevent the robber from moving back on the $\mathcal{R}$ path towards the middle which is essential for the cops in order to capture the robber. These proper *exit-covering* cop combinations are described in the following. For an illustration of the underlying graph structure, we refer to Figure 2.

One cop, say Cop 0, has to be in the $\mathcal{C}$ component of $G_{\hat{n}}$ and the other one (Cop 1) in

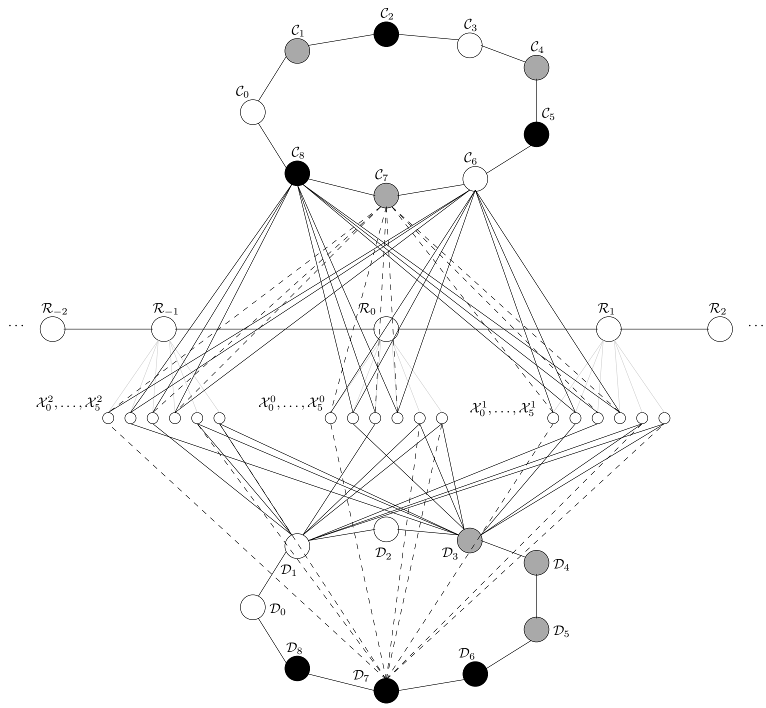

Figure 2 A part of the structure of $G_{\hat{n}}$ around the $\mathcal{X}$ nodes. A $\mathcal{C}$ node and a $\mathcal{D}$ node together cover the exits of an $\mathcal{R}$ node if and only if they have the same color.

the $\mathcal{D}$ component. The nodes in the $\mathcal{C}$ (resp., $\mathcal{D}$) component induce a simple cycle on $G_{\hat{n}}$. Assume for simplicity that the number of nodes in these two cycles are both multiples of 3. Now, we can imagine that the nodes in the $\mathcal{C}$ cycle are consecutively colored $0, 1, 2, 0, 1, 2, \ldots$ and that the nodes in the $\mathcal{D}$ cycle are colored $0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2$, resulting in three equally-sized monochromatic blocks. Now, the nodes in $\mathcal{C}$ and $\mathcal{D}$ are connected to the nodes in $\mathcal{X}$ in so that Cop 0 (in $\mathcal{C}$) and Cop 1 (in $\mathcal{D}$) cover all exits of an $\mathcal{R}$ node if and only if the nodes the two cops are occupying have the same color. Thus, if Cop 0 wants to move, e.g., clockwise, in her $\mathcal{C}$ cycle, then between any two consecutive steps, she has to wait for Cop 1 to travel roughly a third of her $\mathcal{D}$ cycle.

Similarly, using an independent color pallet, we color the nodes along the $\mathcal{R}$ path $3, 4, 5, 3, 4, 5, \ldots$ and color the nodes along the $\mathcal{C}$ cycle $3, \ldots, 3, 4, \ldots, 4, 5, \ldots, 5$. To prevent the robber from moving back towards the middle of her $\mathcal{R}$ path, we construct $G_{\hat{n}}$ so that a $\mathcal{C}$ node covers an $\mathcal{R}$ node if and only if they do *not* have the same color. Thus, if Cop 0 proceeds along her $\mathcal{C}$ cycle, then as soon as the color of the $\mathcal{C}$ node changes, the robber is forced to move one step forward along the $\mathcal{R}$ path. This accounts for Cop 0 traversing roughly a third of the $\mathcal{C}$ cycle for each step of the robber along the $\mathcal{R}$ path. The direction of the robber's movement (towards either end of the $\mathcal{R}$ path) is determined by the direction (clockwise or counterclockwise) of the movement of Cop 0 along the $\mathcal{C}$ cycle. We refer to Figure 3 for an illustration of how the robber is pushed along the $\mathcal{R}$ by a cop residing in $\mathcal{C}$.

Now, we design the graph $G_{\hat{n}}$ so that the $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{R}$ components consist of roughly $\hat{n}$ nodes. Thus, the robber takes $\Omega(\hat{n})$ steps until she is captured, for each of her steps Cop 0 has to take $\Omega(\hat{n})$ steps, and for each step of Cop 0, Cop 1 has to take $\Omega(\hat{n})$ steps, resulting in a total capture time of $\Omega(\hat{n}^3)$. Since every component of the $G_{\hat{n}}$ except $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{D}$ is of constant size, $\hat{n}$ is linear in the number n of nodes. Hence, we obtain a capture time of

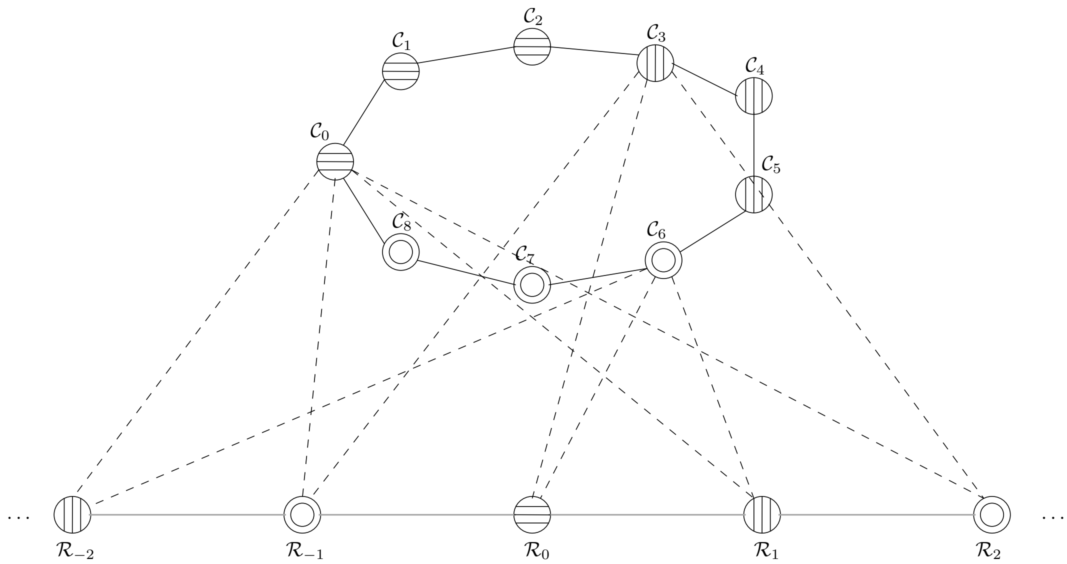

Figure 3 The edge structure between the $\mathcal{C}$ nodes and the $\mathcal{R}$ nodes. If a cop is in a $\mathcal{C}$ node and the robber in an $\mathcal{R}$ node, then the robber has to make sure that that these two nodes have the same color in order to avoid capture in the next move. By circling clockwise in $\mathcal{C}$, the cop can force the robber towards one end of the $\mathcal{R}$ path, by circling counterclockwise towards the other.

$\Omega(n^3)$ for the graph class $\{G_{\hat{n}}\}_{\hat{n}\geq 3, \hat{n}\equiv 0 \pmod 3}$.

Before proceeding to the exact details of the graph construction, four remarks are in order. Firstly, when the robber is flushed out of $\mathcal{U}$ to $\mathcal{R}$, there is an intermediate step between leaving $\mathcal{U}$ and arriving in $\mathcal{R}$ for technical reasons: Flushed out of $\mathcal{U}$, the robber actually has to go to a graph component called $\mathcal{A}$. Then, one cop moves to another graph component denoted $\mathcal{T}$ from which she covers all of $\mathcal{A}$ while ensuring that all nodes in $\mathcal{U}$ the robber could go to are covered. Thus, the robber is flushed out of $\mathcal{A}$ and is forced to go to $\mathcal{R}$ so that we can proceed as explained above.

Secondly, since Cop 0 has to be able to cover nodes from $\mathcal{R}$ when she is in $\mathcal{C}$, there have to be edges between $\mathcal{R}$ and $\mathcal{C}$. To prevent the robber from escaping $\mathcal{R}$ by going to a node in $\mathcal{C}$ via one of these edges, we augment the graph with a special $\mathcal{P}$ node. If, after the robber indeed moves to the $\mathcal{C}$ component, Cop 1 moves to $\mathcal{P}$ and Cop 0 moves to $\mathcal{S}_0$, then together they cover all $\mathcal{C}$ nodes and all their neighbors, ensuring that the robber will be captured in the next round.

Thirdly, the robber does not have to start in $\mathcal{U}$, but in fact it is best for her if she does (in the sense of increasing the capture time against the best cop strategy), provided that she cannot be captured immediately. In turn, this means that the cops should start in $\mathcal{S}_0$ and $\mathcal{S}_1$ in order to force the robber to start elsewhere. Even if the robber starts elsewhere, the cop strategy explained above forces the robber to the $\mathcal{R}$ component. Moreover, the robber can simply start in the middle of the $\mathcal{R}$ path and the cops cannot avoid having to go through the exit-covering routine explained above.

Fourthly, while the cop strategy explained above chases the robber from the middle of the $\mathcal{R}$ path to one of its ends, for simplicity, we will formally present a slightly simplified version where the end of the path the robber is chased to is fixed in advance (so the robber will be chased from one end of the path to the other end in the worst case for the cops).

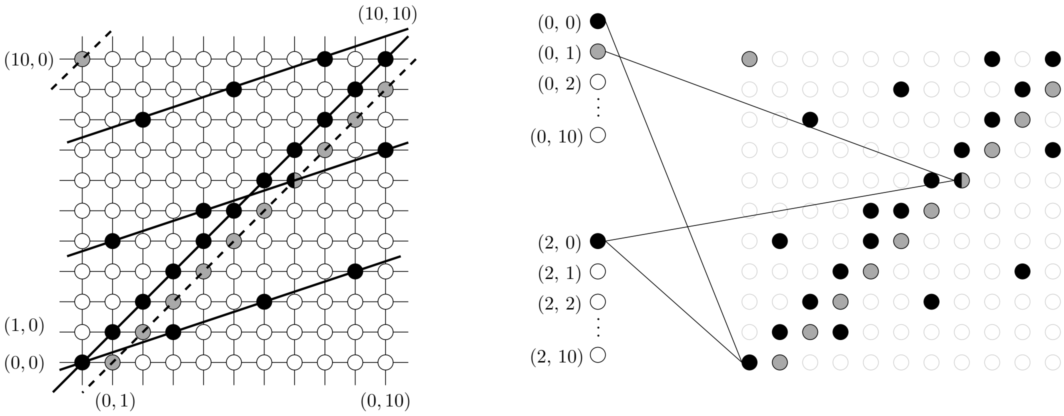

Figure 4 The construction of $\mathcal{U}$. On the left side, we see the 11×11 grid constituting $\mathcal{E}$ and three example lines $\mathcal{L}_{0,0}$, $\mathcal{L}_{0,1}$ and $\mathcal{L}_{2,0}$ from $\mathcal{L}$. On the right side, we see the nodes of the bipartite graph $G_{\mathcal{E},\mathcal{L}}$ where the left-hand column of nodes constitutes (a part of) one side of the bipartition, $\mathcal{L}$, and the right-hand grid the other side, $\mathcal{E}$. The edges between $\mathcal{E}$ and $\mathcal{L}$ are determined by the incidence relation of the nodes and lines in the left-hand 11×11 grid.

3.2 The Graph Construction

As explained above, the graphs $G_{\hat{n}}$ we are about to construct contain a component $\mathcal{U}$ in which the robber cannot be captured and from which the robber can only be flushed out by a specific cop combination outside of $\mathcal{U}$. Hence, the subgraph of any $G_{\hat{n}}$ induced by (the respective) $\mathcal{U}$ cannot be a 2-cop-win graph. As we want to generalize our graph construction to the case of more than 2 cops, we thus need a way to construct a graph where k cops cannot capture the robber. For this, inspired by the use of projective planes for constructing graphs with high cop numbers in [20], we will use incidence graphs of objects resembling affine planes. An explicit construction (for the case of 2 cops) is given in the following. For an illustration of the construction, we refer to Figure 4.

Let $\mathcal{E} = \{\mathcal{E}_{i,j} \mid 0 \leq i \leq 10 \wedge 0 \leq j \leq 10\}$ be a set of *elements* which we can imagine as arrayed in an 11×11 grid. Let $\mathcal{L} = \{\mathcal{L}_{i,j} \mid 0 \leq i \leq 9 \wedge 0 \leq j \leq 10\}$ be a set of *lines* where each $\mathcal{L}_{i,j}$ is defined as $\mathcal{L}_{i,j} = \{\mathcal{E}_{h,h(i+1)+j \pmod{11}} \mid 0 \leq h \leq 10\}$. Thus, each line $\mathcal{L}_{i,j}$ may be considered as a "line modulo 11" in our grid which goes through the element $\mathcal{E}_{0,j}$ and whose slope is determined by the parameter i (or more precisely $i+1$).

Now consider the *incidence graph* $G_{\mathcal{E},\mathcal{L}}$ for $\mathcal{E}$ and $\mathcal{L}$ which is defined as follows: The nodes of $G_{\mathcal{E},\mathcal{L}}$ are exactly the elements and lines defined above, i.e., $V(G_{\mathcal{E},\mathcal{L}}) = \mathcal{E} \cup \mathcal{L}$, and there is an edge between some node $\mathcal{E}_{i,j}$ and some node $\mathcal{L}_{i',j'}$ if and only if $\mathcal{E}_{i,j}$ is contained in the set $\mathcal{L}_{i',j'}$ (i.e., if and only if $\mathcal{E}_{i,j}$ lies on the line $\mathcal{L}_{i',j'}$). There are no other edges, hence $G_{\mathcal{E},\mathcal{L}}$ is bipartite where one side of the bipartition is given by $\mathcal{E}$ and the other side by $\mathcal{L}$.

▶ **Lemma 2.** *In $G_{\mathcal{E},\mathcal{L}}$, any two nodes in $\mathcal{E}$ have at most one common neighbor in $\mathcal{L}$. Also, any two nodes in $\mathcal{L}$ have at most one common neighbor in $\mathcal{E}$.*

▶ **Lemma 3.** *Let $i \in \{0, \dots, 10\}$ be fixed. Any node from $\mathcal{L}$ has exactly one neighbor in $G_{\mathcal{E},\mathcal{L}}$ of the form $\mathcal{E}_{i,j}$ and exactly one neighbor in $G_{\mathcal{E},\mathcal{L}}$ of the form $\mathcal{E}_{j,i}$.*

▶ **Lemma 4.** *Let $i \in \{0, \dots, 9\}$ be fixed. Any node from $\mathcal{E}$ has exactly one neighbor in $G_{\mathcal{E},\mathcal{L}}$ of the form $\mathcal{L}_{i,j}$.*

For the construction of $G_{\hat{n}}$, we will *borrow* nodes from $G_{\mathcal{E},\mathcal{L}}$ and we will assume that the borrowed nodes take along their relationship concerning edges between them, i.e., there is an edge between two borrowed nodes in our graph construction if and only if there is an edge between those two nodes in $G_{\mathcal{E},\mathcal{L}}$.

We construct $G_{\hat{n}}$ as given in the following. The vertex set of $G_{\hat{n}}$ is defined as the (disjoint) union of smaller vertex sets that constitute different parts of the graph with different purposes:

$$
\begin{aligned}
V(G_{\hat{n}}) &= \mathcal{E} \cup \mathcal{L}^* \cup \mathcal{S} \cup \mathcal{A} \cup \mathcal{T} \cup \mathcal{R} \cup \mathcal{C} \cup \mathcal{D} \cup \{\mathcal{P}\} \cup \mathcal{X}, \text{ where} \\
\mathcal{L}^* &= \{\mathcal{L}_{i,j} \mid 0 \le i \le 3 \wedge 0 \le j \le 10\} \\
\mathcal{S} &= \{\mathcal{S}_0, \mathcal{S}_1\} \\
\mathcal{A} &= \{\mathcal{A}_0, \dots, \mathcal{A}_3\} \\
\mathcal{T} &= \{\mathcal{T}_0, \dots, \mathcal{T}_3\} \\
\mathcal{R} &= \{\mathcal{R}_{-\hat{n}}, \dots, \mathcal{R}_{\hat{n}}\} \\
\mathcal{C} &= \{\mathcal{C}_0, \dots, \mathcal{C}_{\hat{n}-1}\} \\
\mathcal{D} &= \{\mathcal{D}_0, \dots, \mathcal{D}_{\hat{n}-1}\} \\
\mathcal{X} &= \mathcal{X}^0 \cup \mathcal{X}^1 \cup \mathcal{X}^2 \\
\mathcal{X}^j &= \{\mathcal{X}^j_0, \dots, \mathcal{X}^j_5\} \text{ for all } 0 \le j \le 2
\end{aligned}
$$

The edges of $G_{\hat{n}}$ are specified in Table 1. Moreover, we set $\mathcal{U} = \mathcal{E} \cup \mathcal{L}^*$. Furthermore, we call the nodes in $\mathcal{X}$ *exits*, and for each node $\mathcal{R}_i \in \mathcal{R}$ we call the nodes from $\mathcal{X}$, that are connected to $\mathcal{R}_i$, the *exits of* $\mathcal{R}_i$.

The node subsets $\mathcal{E}$ and $\mathcal{L}^*$ are borrowed from $G_{\mathcal{E},\mathcal{L}}$, but also the (renamed) nodes in $\mathcal{X}$ are borrowed from $G_{\mathcal{E},\mathcal{L}}$: We consider $\mathcal{X}$ as a subset of $\mathcal{L} \setminus \mathcal{L}^*$. To ensure that no node in $\mathcal{E}$ covers too many exits of some $\mathcal{R}$ node, the exits of any $\mathcal{R}$ node are borrowed (disjointly) from a set of $\mathcal{L} \setminus \mathcal{L}^*$ nodes of the same slope. More precisely,

$$
\begin{aligned}
&\mathcal{X}^0 \text{ is borrowed from } \{\mathcal{L}_{4,j} \mid 0 \le j \le 10\}\ , \\
&\mathcal{X}^1 \text{ is borrowed from } \{\mathcal{L}_{5,j} \mid 0 \le j \le 10\}\ , \\
&\mathcal{X}^2 \text{ is borrowed from } \{\mathcal{L}_{6,j} \mid 0 \le j \le 10\}\ .
\end{aligned}
$$

As long as the above conditions are met, we do not care about the explicit choice of $\mathcal{X}$ as a subset of $\mathcal{L} \setminus \mathcal{L}^*$. We obtain the following corollary from Lemma 2:

▶ **Corollary 5.** *Any two nodes in $\mathcal{E}$ have at most one common neighbor in $\mathcal{L}^*$. Any two nodes in $\mathcal{L}^* \cup \mathcal{X}$ have at most one common neighbor in $\mathcal{E}$.*

3.3 Observations

Before specifying asymptotically best strategies for the robber and the cops in $G_{\hat{n}}$, we gather some useful observations about the structure of $G_{\hat{n}}$. In particular, we examine which cop combinations cover certain neighbors of certain nodes. We start by showing that the cops have to be in $\mathcal{S}_0$ and $\mathcal{S}_1$ in order to flush the robber out of $\mathcal{U}$.

▶ **Lemma 6.** *Consider any $u \in \mathcal{U}$. The only cop combination not containing u that covers all neighbors of u in $\mathcal{U}$ is $\{\mathcal{S}_0, \mathcal{S}_1\}$.*

We proceed by showing that if the robber has been flushed out of $\mathcal{U}$ to some node $\mathcal{A}_i$, then the cops can only make progress by going to $\{\mathcal{S}_0, \mathcal{T}_i\}$ because otherwise the robber can go back to $\mathcal{U}$ (or there is no progress if the cops simply stay in $\{\mathcal{S}_0, \mathcal{S}_1\}$).

Node	Neighbors
$\mathcal{E}_{i,j}$	$\mathcal{L}^*$ nodes as determined by $G_{\mathcal{E},\mathcal{L}}$ $\mathcal{S}_{i \pmod 2}$ $\mathcal{A}_{j \pmod 4}$ $\mathcal{T}_{j \pmod 4}$ if $i \pmod 2 = 1$ $\mathcal{X}$ nodes as determined by $G_{\mathcal{E},\mathcal{L}}$
$\mathcal{L}_{i,j}$	$\mathcal{E}$ nodes as determined by $G_{\mathcal{E},\mathcal{L}}$ $\mathcal{S}_{i \pmod 2}$ $\mathcal{A}_{i \pmod 4}$ $\mathcal{T}_{i \pmod 4}$

Node	Neighbors
$\mathcal{S}_0$	$\mathcal{C}_i$ for all i $\mathcal{X}_i^j$, where $j \in \{0,1,2\}$ and $i \in \{0,1,2\}$ every $\mathcal{E}_{i,j}$ with $i \pmod 2 = 0$ every $\mathcal{L}_{i,j}$ with $i \pmod 2 = 0$
$\mathcal{S}_1$	$\mathcal{T}_i$ for all i $\mathcal{X}_i^j$, where $j \in \{0,1,2\}$ and $i \in \{3,4,5\}$ every $\mathcal{E}_{i,j}$ with $i \pmod 2 = 1$ every $\mathcal{L}_{i,j}$ with $i \pmod 2 = 1$
$\mathcal{A}_i$	$\mathcal{T}_j$ for all j $\mathcal{R}_0$ $\mathcal{D}_j$ for all j $\mathcal{E}_{j,h}$ for all $h \pmod 4 = i$ $\mathcal{L}_{j,h}$ for all $j \pmod 4 = i$
$\mathcal{T}_i$	$\mathcal{S}_1$ $\mathcal{A}_j$ for all j $\mathcal{T}_j$ for all $j \neq i$ $\mathcal{D}_j$ for all j $\mathcal{P}$ $\mathcal{X}_h^j$, where $j \in \{0,1,2\}$ and $h \in \{3,4,5\}$ $\mathcal{E}_{j,h}$ for all $j \pmod 2 = 1$ and $h \pmod 4 = i$ $\mathcal{L}_{j,h}$ for all $j \pmod 4 = i$

Node	Neighbors
$\mathcal{R}_i$	$\mathcal{A}_j$ for all j if $i = 0$ $\mathcal{R}_j$ for $j = i-1$ and $j = i+1$ $\mathcal{C}_j$ for all $0 \leq j \leq \hat{n}/3$ if $i \pmod 3 = 0$ $\mathcal{C}_j$ for all $\hat{n}/3 \leq j \leq 2\hat{n}/3$ if $i \pmod 3 = 1$ $\mathcal{C}_0$ and $\mathcal{C}_j$ for all $2\hat{n}/3 \leq j \leq \hat{n}-1$ if $i \pmod 3 = 2$ $\mathcal{P}$ $\mathcal{X}_h^j$ where $j = i \pmod 3$ and $h \in \{0,1,2,3,4,5\}$
$\mathcal{C}_i$	$\mathcal{S}_0$ $\mathcal{R}_j$ for all $j \pmod 3 = 0$ if $0 \leq i \leq \hat{n}/3$ $\mathcal{R}_j$ for all $j \pmod 3 = 1$ if $\hat{n}/3 \leq i \leq 2\hat{n}/3$ $\mathcal{R}_j$ for all $j \pmod 3 = 2$ if $i = 0$ or $2\hat{n}/3 \leq i \leq \hat{n}-1$ $\mathcal{C}_j$ for $j \equiv i-1 \pmod{\hat{n}}$ and $j \equiv i+1 \pmod{\hat{n}}$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{0,1,3\}$ if $i \pmod 3 = 0$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{0,2,3\}$ if $i \pmod 3 = 1$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{1,2,3\}$ if $i \pmod 3 = 2$
$\mathcal{D}_i$	$\mathcal{A}_j$ for all j $\mathcal{T}_j$ for all j $\mathcal{D}_j$ for $j \equiv i-1 \pmod{\hat{n}}$ and $j \equiv i+1 \pmod{\hat{n}}$ $\mathcal{P}$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{2,4,5\}$ if $0 \leq i \leq \hat{n}/3-1$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{1,4,5\}$ if $\hat{n}/3 \leq i \leq 2\hat{n}/3-1$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{0,4,5\}$ if $2\hat{n}/3 \leq i \leq \hat{n}-1$
$\mathcal{P}$	$\mathcal{T}_i$ for all i $\mathcal{R}_i$ for all i $\mathcal{D}_i$ for all i $\mathcal{X}_i^j$ where $j \in \{0,1,2\}$ and $i = 3$

Table 1 A listing of the edges of $G_{\hat{n}}$. In each block, the nodes listed in the right column are the neighboring nodes of the node listed in the left column. For simplicity, we omit a separate block for specifying the neighbors of the $\mathcal{X}$ nodes. They can be inferred from the other blocks. The $\mathcal{X}$ nodes do not have any edges connecting them to each other since they are borrowed from the $\mathcal{L}$ part of $G_{\mathcal{E},\mathcal{L}}$.

▶ **Lemma 7.** *Consider any $\mathcal{A}_i$. The only cop combinations not containing $\mathcal{A}_i$ that cover all neighbors of $\mathcal{A}_i$ in $\mathcal{U}$ are $\{\mathcal{S}_0, \mathcal{S}_1\}$ and $\{\mathcal{S}_0, \mathcal{T}_i\}$.*

The following lemma shows that if the cops allow the robber to go to an exit of some $\mathcal{R}$ node, then they have to go back to $\{\mathcal{S}_0, \mathcal{S}_1\}$ in order to prevent the robber from going to $\mathcal{U}$.

▶ **Lemma 8.** *Consider any $\mathcal{X}_i^j$. The only cop combination not containing $\mathcal{X}_i^j$ that covers all neighbors of $\mathcal{X}_i^j$ in $\mathcal{U}$ is $\{\mathcal{S}_0, \mathcal{S}_1\}$.*

As Lemma 8 already indicates, the cops do not want the robber to be able to go to an exit from an $\mathcal{R}$ node. The next lemma characterizes the cop combinations from where they can prevent the robber from doing that.

▶ **Lemma 9.** *Consider any $\mathcal{R}_i$. The only cop combinations not containing any $\mathcal{R}_j$ with $j \equiv i \pmod 3$ that cover all exits of $\mathcal{R}_i$ are $\{\mathcal{S}_0, \mathcal{S}_1\}$, $\{\mathcal{S}_0, \mathcal{T}_j\}$ for any j, and $\{\mathcal{C}_j, \mathcal{D}_h\}$ for any pair (j, h) satisfying one of the following three conditions:*

1. *$j \pmod 3 = 0$ and $0 \leq h \leq \hat{n}/3 - 1$,*
2. *$j \pmod 3 = 1$ and $\hat{n}/3 \leq h \leq 2\hat{n}/3 - 1$,*
3. *$j \pmod 3 = 2$ and $2\hat{n}/3 \leq h \leq \hat{n} - 1$.*

$\{c_0(t), c_1(t)\}$	$r(t-1)$	$r(t)$
$\neq \{\mathcal{S}_0, \mathcal{S}_1\}$	some node in $\mathcal{U}$	some uncovered node in $\mathcal{U}$
$\{\mathcal{S}_0, \mathcal{S}_1\}$	some node in $\mathcal{U}$	some node in $\mathcal{A}$
$\{\mathcal{S}_0, \mathcal{S}_1\}$ or $\{\mathcal{S}_0, \mathcal{T}_i\}$	$\mathcal{A}_i$	$\mathcal{R}_0$
$\neq \{\mathcal{S}_0, \mathcal{S}_1\}$ and $\neq \{\mathcal{S}_0, \mathcal{T}_i\}$	$\mathcal{A}_i$	some uncovered node in $\mathcal{U}$
not covering all exits of $\mathcal{R}_i$	$\mathcal{R}_i$	some uncovered exit of $\mathcal{R}_i$
covering all exits of $\mathcal{R}_i$	$\mathcal{R}_i$	the uncovered node from $\{\mathcal{R}_{i-1}, \mathcal{R}_i, \mathcal{R}_{i+1}\}$ with smallest absolute index; if all are covered, stay in $\mathcal{R}_i$
$\neq \{\mathcal{S}_0, \mathcal{S}_1\}$	$\mathcal{X}_i^j$	some uncovered node in $\mathcal{U}$
$\{\mathcal{S}_0, \mathcal{S}_1\}$	$\mathcal{X}_i^j$	some node from $\{\mathcal{R}_{-1}, \mathcal{R}_0, \mathcal{R}_1\}$

Table 2 The robber's strategy

Observe that the cop combinations from Lemma 9 are independent of the choice of the considered $\mathcal{R}_i$ which implies that these cop combinations cover all nodes in $\mathcal{X}$. We call such a cop combination *exit-blocking*. Furthermore, we call an exit-blocking cop combination *proper* if it does not contain a node from $\mathcal{S}$ (i.e., it consists of a node from $\mathcal{C}$ and a node from $\mathcal{D}$). Lastly, we call a proper exit-blocking cop combination $\{\mathcal{C}_i, \mathcal{D}_j\}$ *forcing* if there exist $h, h' \in \{-1, 0, 1\}$, $h \neq h'$, such that the cops cover all $\mathcal{R}_{h \pmod 3}$ and all $\mathcal{R}_{h' \pmod 3}$. A close look at the construction of $G_{\hat{n}}$ shows that a proper exit-blocking cop combination $\{\mathcal{C}_i, \mathcal{D}_j\}$ is forcing if and only if $i \in \{0, \hat{n}/3, 2\hat{n}/3\}$.

Proper exit-blocking cop combinations prevent the robber from going back (too much) towards the middle of the $\mathcal{R}$ path since they contain a $\mathcal{C}$ node which by its nature is connected to every third $\mathcal{R}$ node. Thus, in order to be able to chase the robber towards one end of the $\mathcal{R}$ path, the cops have to stay in proper exit-blocking cop combinations.

The $\mathcal{C}$ node in a forcing proper exit-covering cop combination covers more $\mathcal{R}$ nodes than the $\mathcal{C}$ node in a usual proper exit-covering cop combination and thereby forces the robber to move one step towards the end of her $\mathcal{R}$ path. In order to chase the robber another step, the cops have to go to a forcing proper exit-covering cop combination containing a different $\mathcal{C}$ node. The following lemma shows a lower bound on the time it takes the cops to go from one forcing proper exit-covering cop combination to another with a different $\mathcal{C}$ node, while using only proper exit-covering cop combinations on the way. Refer to Figures 2 and 3 for illustrations of the underlying idea.

▶ **Lemma 10.** *Let* $(\{\mathcal{C}_i, \mathcal{D}_j\} = \{c_0(t), c_1(t)\}, \{c_0(t+1), c_1(t+1)\}, \ldots, \{c_0(t+h), c_1(t+h)\} = \{\mathcal{C}_{i'}, \mathcal{D}_{j'}\})$ *be a sequence of proper exit-blocking cop combinations describing the combined movement of the two cops from time* t *to time* $t+h$*. If* $\{\mathcal{C}_i, \mathcal{D}_j\}$ *and* $\{\mathcal{C}_{i'}, \mathcal{D}_{j'}\}$ *are forcing and* $i \neq i'$*, then* $h \geq \hat{n}/3 \cdot (\hat{n}/3 - 1) \in \Omega(\hat{n}^2)$*.*

3.4 The Robber's Strategy

Here, we explicitly specify a strategy for the robber that ensures that 2 cops need time $\Omega(n^3)$ to capture the robber in $G_{\hat{n}}$:

If the cops are in $\mathcal{S}_0$ and $\mathcal{S}_1$ in round 0, then the robber starts in $\mathcal{R}_0$, otherwise the robber starts in some node in $\mathcal{U}$ that is not covered by any of the cops (which exists by Lemma 6). Depending on where the cops are, the robber moves as specified in Table 2 (as long as she is not captured yet).

We show now that the specified strategy is well-defined, i.e., that the robber can perform any step in the strategy and that no other situations than the specified ones can occur if the

$r(t-1)$	$(c_0(t-1), c_1(t-1))$	$(c_0(t), c_1(t))$
$\mathcal{A}_i$	$(\mathcal{S}_0, \mathcal{S}_1)$	$(\mathcal{S}_0, \mathcal{T}_i)$
$\neq \mathcal{A}_i$ for all i	$(\mathcal{S}_0, \mathcal{S}_1)$	$(\mathcal{S}_0, \mathcal{T}_j)$ for some j
arbitrary	$(\mathcal{S}_0, \mathcal{T}_i)$	$(\mathcal{C}_0, \mathcal{D}_0)$
$\neq \mathcal{C}_h$ for all h	$(\mathcal{C}_i, \mathcal{D}_j)$	$\big(\mathcal{C}_{i+1 \pmod{\hat{n}}}, \mathcal{D}_{j+1 \pmod{\hat{n}}}\big)$ if this covers all nodes in $\mathcal{X}$ $\big(\mathcal{C}_i, \mathcal{D}_{j+1 \pmod{\hat{n}}}\big)$ otherwise
$\mathcal{C}_h$	$(\mathcal{C}_i, \mathcal{D}_j)$	$(\mathcal{S}_0, \mathcal{P})$

Table 3 The cops' strategy

robber follows the strategy. For the first part, we go through the table line by line:

By Lemma 6, if the robber is in some node $u \in \mathcal{U}$, then she can always go to some uncovered node in $\mathcal{U}$, provided the cops are not in $\mathcal{S}_0$ and $\mathcal{S}_1$. She can also go from u to some node in $\mathcal{A}$ since any node in $\mathcal{U}$ has some node in $\mathcal{A}$ as a neighbor, by the construction of $G_{\hat{n}}$. Similarly, the robber can go from any node in $\mathcal{A}$ to $\mathcal{R}_0$. By Lemma 7, if the robber is in some node $\mathcal{A}_i$, then she can always go to some uncovered node in $\mathcal{U}$, provided the cops are not in $\mathcal{S}_0$ and $\mathcal{S}_1$ or in $\mathcal{S}_0$ and $\mathcal{T}_i$. The instructions where to go to from $\mathcal{R}_i$ are trivially satisfiable. From $\mathcal{X}_i^j$, the robber can always go to some uncovered node in $\mathcal{U}$ if the cops are not in $\mathcal{S}_0$ and $\mathcal{S}_1$, by Lemma 8. She can also go to either $\mathcal{R}_{-1}$, $\mathcal{R}_0$ or $\mathcal{R}_1$ from $\mathcal{X}_i^j$ since each $\mathcal{X}_i^j$ is connected to exactly one of those three $\mathcal{R}$ nodes, by the construction of $G_{\hat{n}}$.

Moreover since the robber starts in a node in $\mathcal{U}$ or $\mathcal{R}$, Table 2 covers all situations where the robber is in some node in $\mathcal{U}$, $\mathcal{A}$, $\mathcal{R}$ or $\mathcal{X}$, and each instruction ends with the robber being in one of those nodes, the presented strategy specifies what the robber has to do for every possibly occurring situation.

3.5 The Cops' Strategy

Now, we explicitly specify a strategy for the cops that ensures that the robber is captured in time $\mathcal{O}(n^3)$ in $G_{\hat{n}}$:

Cop 0 starts in $\mathcal{S}_0$ and Cop 1 starts in $\mathcal{S}_1$ in round 0. Depending on where the robber is, the cops move as specified in Table 3. There is one exception however: If a cop can capture the robber immediately, then she does so, overriding any possible instruction from the table.

We show now that the specified strategy is well-defined, i.e., that the cops can actually perform any step in the strategy and that no other situations than the specified ones can actually occur[2] if the cops follow the strategy:

The construction of $G_{\hat{n}}$ ensures that the cops can actually move from the cop combinations at time $t-1$ given in Table 3 to the cop combinations at time t. Since the cops start in $\mathcal{S}_0$ and $\mathcal{S}_1$, the only thing that is left to show is that from $(\mathcal{S}_0, \mathcal{P})$ (which is the only output combination that is not dealt with on the input side) the cops can capture the robber at time $t+1$, provided that the robber is in some $\mathcal{C}_h$ at time $t-1$. For that, it is sufficient to observe that any neighbor of $\mathcal{C}_h$, and $\mathcal{C}_h$ itself, is covered by $\mathcal{S}_0$ or $\mathcal{P}$.

3.6 A Lower Bound for the Robber's Strategy

Here, we show that the strategy for the robber specified in Table 2 ensures that the cops need time $\Omega(n^3)$ to capture the robber in $G_{\hat{n}}$. For convenience, we assume throughout the

[2] More precisely, if a situation occurs that is not specified in Table 3, then the cops can capture the robber immediately.

following lower bound considerations that if a cop can capture the robber immediately, then she does so. This certainly cannot worsen any strategy the cops follow.

We start by observing that the set of $\mathcal{R}$ nodes can be partitioned into three roughly equally-sized sets such that the $\mathcal{R}$ nodes in each such set have exactly the same exits. As the following lemma shows, if the robber is in an $\mathcal{R}$ node, then she does not need to worry about a cop being in another $\mathcal{R}$ node that has (and therefore covers) the same exits, since such a situation cannot occur if the robber follows the specified strategy.

We proceed by determining the nodes the robber can be captured in. Then, using Lemma 11 and Lemma 12, we give a lower bound on the capture time of $G_{\hat{n}}$.

▶ **Lemma 11.** *If the robber follows the strategy specified in Section 3.4, then the following holds: If the robber is in some node $\mathcal{R}_i$ at time t and is not captured at time $t+1$, then neither of the 2 cops can be in some node $\mathcal{R}_j$ with $j \equiv i \pmod 3$ at time $t+1$.*

▶ **Lemma 12.** *If the robber follows the strategy specified in Section 3.4, then she can only be captured in $\mathcal{R}_{\hat{n}}$ or $\mathcal{R}_{-\hat{n}}$.*

▶ **Lemma 13.** *If the robber follows the strategy specified in Section 3.4, then 2 cops need time $\Omega(\hat{n}^3)$ to capture the robber in $G_{\hat{n}}$.*

3.7 An Upper Bound for the Cops' Strategy

While the aim of this work is a lower bound, we need to show that 2 cops can actually capture the robber in $G_{\hat{n}}$, in order to use $G_{\hat{n}}$ as a lower bound graph for the capture time for 2 cops in 2-cop-win graphs. We start by showing that from a proper exit-blocking cop combination the cops can always go to another proper exit-blocking cop combination by doing one of the following: Both cops move to the next node in their respective cycle or only the cop in the $\mathcal{D}$ cycle moves to the next node.

▶ **Lemma 14.** *If $(\mathcal{C}_i, \mathcal{D}_j)$ is an exit-blocking cop combination, then it holds that at least one of $(\mathcal{C}_i, \mathcal{D}_{j+1 \pmod{\hat{n}}})$ and $(\mathcal{C}_{i+1 \pmod{\hat{n}}}, \mathcal{D}_{j+1 \pmod{\hat{n}}})$ is an exit-blocking cop combination.*

The following lemma shows that, once the cops reach $\mathcal{C}_0$ and $\mathcal{D}_0$, the robber cannot ever leave $\mathcal{R}$ without being captured in the next two moves. Then, using Lemma 14 and Lemma 15, we give an upper bound on the capture time of $G_{\hat{n}}$.

▶ **Lemma 15.** *Let $r(t) \in \mathcal{R}$ and $(c_0(t+1), c_1(t+1)) = (\mathcal{C}_0, \mathcal{D}_0)$ for some point in time t. If the robber leaves $\mathcal{R}$ at some later point in time t', i.e., if $r(t') \notin \mathcal{R}$ for some $t' > t$, then the robber will be captured at time $t'' \leq t' + 2$, provided the two cops follow the strategy specified in Section 3.5.*

▶ **Lemma 16.** *If the two cops follow the strategy specified in Section 3.5, then they capture the robber in time $\mathcal{O}(\hat{n}^3)$ in $G_{\hat{n}}$.*

Finally, by Lemma 13 we get that for the case of 2 cops, the capture time of the graph family $\{G_{\hat{n}}\}_{\hat{n} \geq 3, \hat{n} \equiv 0 \pmod 3}$ is $\Omega(\hat{n}^3) \subseteq \Omega(n^3)$ and by Lemma 16 we get that every graph in $\{G_{\hat{n}}\}_{\hat{n} \geq 3, \hat{n} \equiv 0 \pmod 3}$ is 2-cop-win. Together, these lemmas yield Theorem 1 for 2 cops.

The Case of $k > 2$.

Our graph construction and the corresponding lower bound proofs follow closely the design of the case of two cops. To accommodate a third cop, Cop 2, we essentially copy the $\mathcal{D}$ component and ensure, that for every step of Cop 1, Cop 2 has to perform $\Omega(\hat{n})$ steps. For the case of $k > 3$ cops, we simply apply this trick inductively. Due to space limitations, we defer the detailed discussion of the case of $k > 2$ cops to Appendix B.

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A Deferred Proofs

Proof of Lemma 2. Assume for a contradiction that there are two different nodes $\mathcal{E}_{i,j}$ and $\mathcal{E}_{i',j'}$ that have two different common neighbors, say $\mathcal{L}_{h,q}$ and $\mathcal{L}_{h',q'}$. By the definitions of the nodes in $\mathcal{L}$ and $G_{\mathcal{E},\mathcal{L}}$, it follows that $j = i(h+1) + q \pmod{11}$, $j = i(h'+1) + q' \pmod{11}$, $j' = i'(h+1) + q \pmod{11}$ and $j' = i'(h'+1) + q' \pmod{11}$. We obtain $(i'-i)(h+1) \equiv j'-j \equiv (i'-i)(h'+1) \pmod{11}$ which implies $(i'-i)(h'-h) \equiv 0 \pmod{11}$. Since 11 is prime, it follows that $i' = i$ or $h' = h$.

If $h' = h$, then our initial equations imply that also $q' = q$ which contradicts the fact that $\mathcal{L}_{h,q}$ and $\mathcal{L}_{h',q'}$ are different nodes. If $i' = i$, then our initial equations imply that also $j' = j$ which contradicts the fact that $\mathcal{E}_{i,j}$ and $\mathcal{E}_{i',j'}$ are different nodes. The obtained contradiction shows both statements given in the lemma. ◀

Proof of Lemma 3. Let $\mathcal{L}_{q,q'} \in \mathcal{L}$. By the definition of $\mathcal{L}_{q,q'}$, the neighbors of $\mathcal{L}_{q,q'}$ in $G_{\mathcal{E},\mathcal{L}}$ are exactly the nodes from $\{\mathcal{E}_{h,h(q+1)+q' \pmod{11}} \mid 0 \leq h \leq 10\}$. The first lemma statement follows immediately.

For the second statement, assume for a contradiction that $h(q+1)+q' \equiv h'(q+1)+q' \pmod{11}$ for some $0 \leq h, h' \leq 10$ satisfying $h \neq h'$. It follows that $(h'-h)(q+1) \equiv 0 \pmod{11}$, and since 11 is prime and $h \neq h'$, we obtain $(q+1) \equiv 0 \pmod{11}$. But $\mathcal{L}_{q,q'} \in \mathcal{L}$ implies $0 \leq q \leq 9$, yielding a contradiction. ◀

Proof of Lemma 4. The neighbors of some node $\mathcal{E}_{j,h} \in \mathcal{E}$ in $G_{\mathcal{E},\mathcal{L}}$ are exactly the nodes $\mathcal{L}_{q,q'}$ that satisfy $h = j(q+1) + q' \pmod{11}$ which, in turn, are exactly the nodes from $\{\mathcal{L}_{q,h-j(q+1) \pmod{11}} \mid 0 \leq q \leq 9\}$. The lemma statement follows. ◀

Proof of Lemma 6. Assume that no cop is in u. We start by observing that each node $v \in \mathcal{U}$, $v \neq u$, covers at most 1 neighbor of u in $\mathcal{U}$, by Corollary 5 and the bipartiteness of $G_{\mathcal{E},\mathcal{L}}$. Now we consider two cases:

First, assume that $u \in \mathcal{E}$. Then, all neighbors of u in $\mathcal{U}$ are in $\mathcal{L}^*$ and, more specifically, for any $0 \leq i \leq 3$, u has exactly one $\mathcal{L}^*$ node with first index i as a neighbor, by Lemma 4. Thus, according to Table 1, u's neighbors can only be covered by nodes from $\mathcal{U}$, $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{T}$. Moreover, each node from $\mathcal{S}$ covers exactly 2 of u's 4 neighbors in $\mathcal{L}^*$, while each node from $\mathcal{A}$ and each node from $\mathcal{T}$ cover exactly 1 of u's neighbors. By combining these insights with our first observation and the fact that there are only 2 cops to cover u's 4 neighbors in $\mathcal{L}^*$, we see that the cops have to be in $\mathcal{S}_0$ and $\mathcal{S}_1$ in order to cover all neighbors of u in $\mathcal{U}$.

Second, assume instead that $u \in \mathcal{L}^*$. Then, all neighbors of u in $\mathcal{U}$ are in $\mathcal{E}$ and, more specifically, u has exactly 11 neighbors $\mathcal{E}_{i,j}$ where each number from $\{0,\ldots,10\}$ occurs exactly once as the first index and exactly once as the second index, by Lemma 3. Thus, according to Table 1, u's neighbors can only be covered by nodes from $\mathcal{U}$, $\mathcal{S}$, $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{X}$. Moreover, each node from $\mathcal{S}$ covers at most 6 of u's neighbors, while each node from $\mathcal{A}$ and each node from $\mathcal{T}$ cover at most 3 of u's neighbors. Furthermore, since the nodes in $\mathcal{X}$ and the nodes in $\mathcal{L}^*$ are both borrowed from $\mathcal{L}$, Lemma 2 ensures that each node from $\mathcal{X}$ covers at most 1 of u's neighbors. Since we have only two cops to cover the 11 neighbors of u in $\mathcal{U}$, they must be in $\mathcal{S}_0$ and $\mathcal{S}_1$.

Note that the cop combination $(\mathcal{S}_0, \mathcal{S}_1)$ indeed covers all neighbor's of u in $\mathcal{U}$ in both cases, since $(\mathcal{S}_0, \mathcal{S}_1)$ covers the whole of $\mathcal{U}$, according to Table 1. ◀

Proof of Lemma 7. Assume that no cop is in $\mathcal{A}_i$. Consider the set $B_i = \{\mathcal{E}_{0,i}, \ldots, \mathcal{E}_{3,i}\}$ which is a subset of the set of neighbors of $\mathcal{A}_i$ in $\mathcal{U}$, by Table 1. We show a stronger version

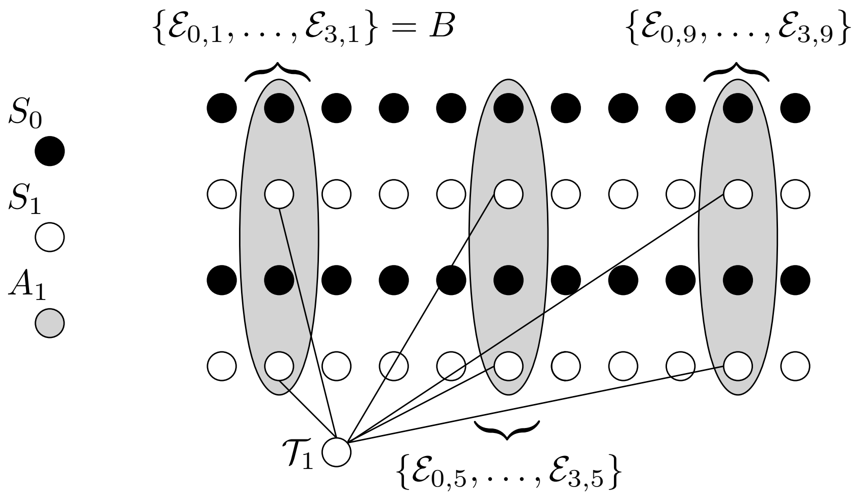

Figure 5 A part of the structure around nodes in $\mathcal{E}$. Each black node is connected to $\mathcal{S}_0$, each white node is connected to $\mathcal{S}_1$ and each node in a grey area is connected to $\mathcal{A}_1$. Each white node in a grey area is connected to $\mathcal{T}_1$. It follows that $(\mathcal{S}_0, \mathcal{S}_1)$ and $(\mathcal{S}_0, \mathcal{T}_1)$ cover all neighbors of $\mathcal{A}_1$ in $\mathcal{E}$, including B_1.

of the lemma statement where we only consider the neighbors of $\mathcal{A}_i$ contained in B_i. For an illustration of the following considerations, we refer to Figure 5.

As can be seen from Table 1, for any $j \neq i$, $\mathcal{A}_j$ and $\mathcal{T}_j$ cover 0 nodes from B_i. Moreover, by Lemma 3, each node from $\mathcal{L}^*$ and each node from $\mathcal{X}$ cover at most 1 node from B_i since the nodes in $\mathcal{L}^*$ and the nodes in $\mathcal{X}$ are both borrowed from $\mathcal{L}$. The only remaining neighbors of nodes in $B_i \subset \mathcal{E}$ are $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{T}_i$, according to Table 1. These three nodes cover exactly 2 nodes from B_i each and $\mathcal{S}_0$ is the only node of the three that covers $\mathcal{E}_{0,i}$.

Since we have only 2 cops to cover the 4 nodes from B_i, we can conclude that the only possible cop combinations for covering all neighbors of $\mathcal{A}_i$ in $\mathcal{U}$ are $\{\mathcal{S}_0, \mathcal{S}_1\}$ and $\{\mathcal{S}_0, \mathcal{T}_i\}$. A close look at Table 1 shows that $\mathcal{S}_0$ covers exactly the neighbors $\mathcal{E}_{j,h}$, resp. $\mathcal{L}_{j,h}$, of $\mathcal{A}_i$ that satisfy $j \pmod 2 = 0$ while $\mathcal{S}_1$ and $\mathcal{T}_i$ both do the same for the neighbors that satisfy $j \pmod 2 = 1$. Thus, $\{\mathcal{S}_0, \mathcal{S}_1\}$ and $\{\mathcal{S}_0, \mathcal{T}_i\}$ are indeed both cop combinations that cover all neighbors of $\mathcal{A}_i$ in $\mathcal{U}$. ◀

Proof of Lemma 8. Since any node in $\mathcal{X}$ is "the same" as a node from $\mathcal{L}^*$ in the sense that it is a node borrowed from $\mathcal{L}$, we can actually use the same proof as given for the second case in (the proof of) Lemma 6. The lemma statement follows. ◀

Proof of Lemma 9. Assume that no cop is in some $\mathcal{R}_j$, where $j \equiv i \pmod 3$. As can be seen from Table 1, each node $\neq \mathcal{P}$ that is not contained in $\mathcal{E}$, $\mathcal{R}$, or $\mathcal{X}$, and covers at least one exit, covers exactly 3 of the 6 exits of $\mathcal{R}_i$ (which are $\mathcal{X}_0^j, \ldots, \mathcal{X}_5^j$ where $j = i \pmod 3$). Since $\mathcal{P}$, any $\mathcal{E}$ node (cf. Lemma 4), any $\mathcal{X}$ node and any $\mathcal{R}_j$, where $j \not\equiv i \pmod 3$, cover less than 3 of those exits, the only possibility for the 2 cops to cover all 6 exits is to be in a cop combination where the two nodes cover exits in a complementary fashion. Now, checking Table 1, we can confirm that these "complementary" cop combinations are exactly those given in the lemma statement. ◀

Proof of Lemma 10. Assume that $\{\mathcal{C}_i, \mathcal{D}_j\}$ and $\{\mathcal{C}_{i'}, \mathcal{D}_{j'}\}$ are forcing and $i \neq i'$. Assume further w.l.o.g. that the cop staying in the $\mathcal{C}$ nodes is Cop 0 and the cop staying in the $\mathcal{D}$ nodes is Cop 1. Note that the definition of proper exit-blocking cop combinations ensures that one cop has to stay in the $\mathcal{C}$ nodes and the other one in the $\mathcal{D}$ nodes from time t to time $t+h$. Note further, that the $\mathcal{C}$ nodes together with the connecting edges form a cycle and the same holds for the $\mathcal{D}$ nodes. We say that a cop moves *clockwise* on the $\mathcal{C}$ or $\mathcal{D}$ cycle, if the index is increased by 1 (modulo $\hat{n}$) when moving from a node to its neighbor, and *counterclockwise* if she moves in the opposite direction.

By our observations about forcing cop combinations following Lemma 9, we know that $i, i' \in \{0, \hat{n}/3, 2\hat{n}/3\}$ which implies that Cop 0 has to take at least $\hat{n}/3$ steps in the $\mathcal{C}$ cycle to get from $\mathcal{C}_i$ to $\mathcal{C}_{i'}$. More precisely, Cop 0 has to take at least $\hat{n}/3$ clockwise steps or at least $\hat{n}/3$ counterclockwise steps; assume that the steps are clockwise (the proof for the counterclockwise case is analogous).

We will show now that for Cop 0, the time between entering some node $\mathcal{C}_q$ from its counterclockwise neighbor $\mathcal{C}_{q-1 \pmod{\hat{n}}}$ and leaving $\mathcal{C}_q$ towards its clockwise neighbor $\mathcal{C}_{q+1 \pmod{\hat{n}}}$ is at least $\hat{n}/3$ (during the sequence of proper exit-blocking cop combinations from time t to time $t+h$).

Assume that $q \pmod 3 = 1$ which implies $q-1 \pmod 3 = 0$ and $q+1 \pmod 3 = 2$. By the definition of proper exit-covering cop combinations (cf. Lemma 9), the following holds: When Cop 0 enters $\mathcal{C}_q$ from $\mathcal{C}_{q-1 \pmod{\hat{n}}}$, then Cop 1 must have been in $\mathcal{D}_{\hat{n}/3-1}$ and move from there to $\mathcal{D}_{\hat{n}/3}$ in order for the two consecutive cop combinations to be proper exit-covering. Similarly, when Cop 0 leaves $\mathcal{C}_q$ towards $\mathcal{C}_{q+1 \pmod{\hat{n}}}$ then Cop 1 must have been in $\mathcal{D}_{2\hat{n}/3-1}$ and move from there to $\mathcal{D}_{2\hat{n}/3}$. Now in order to get from $\mathcal{D}_{\hat{n}/3}$ to $\mathcal{D}_{2\hat{n}/3}$ (via nodes in $\mathcal{D}$) Cop 1 needs at least $\hat{n}/3$ steps. If $q \pmod 3 = 0$ or $q \pmod 3 = 2$, the bound of $\hat{n}/3$ steps is proved analogously.

Since Cop 0 has to enter and leave at least $\hat{n}/3-1$ nodes on her way from $\mathcal{C}_i$ to $\mathcal{C}_{i'}$ and between Cop 0 entering and leaving such a node, Cop 1 has to take at least $\hat{n}/3$ steps, the lemma statement follows.

◀

Proof of Lemma 11. Assume for a contradiction w.l.o.g. that $r(t) = \mathcal{R}_i$, $c_0(t+1) = \mathcal{R}_j$ where $j \equiv i \pmod 3$. and that the robber is not captured at time $t+1$. Assume further that t is the smallest time for which such i and j exist. Now, if $c_0(t) \in \mathcal{A} \cup \mathcal{C} \cup \{\mathcal{P}\} \cup \mathcal{X}$, then Cop 0 would have captured the robber at time $t+1$, by Table 1 and the fact that $j \equiv i \pmod 3$. Thus, again by Table 1, $c_0(t) \in \{\mathcal{R}_{j-1}, \mathcal{R}_j, \mathcal{R}_{j+1}\}$.

Given that $c_0(t) \in \mathcal{R}$ and $r(t) \in \mathcal{R}$, we get from Table 2 that $r(t-1) \in \mathcal{R}$. Let i' be the corresponding index, i.e., $r(t-1) = \mathcal{R}_{i'}$. Again, looking at Table 2 and keeping in mind that $r(t) = \mathcal{R}_i$, we conclude that $\{c_0(t), c_1(t)\}$ covers all exits of $\mathcal{R}_{i'}$. Since $c_0(t) \in \{\mathcal{R}_{j-1}, \mathcal{R}_j, \mathcal{R}_{j+1}\}$, we obtain that there is some $\mathcal{R}_{j'} \in \{c_0(t), c_1(t)\}$ with $j' \equiv i' \pmod 3$, by Lemma 9. This yields a contradiction to the minimality of t. ◀

Proof of Lemma 12. The choice of the starting node of the robber ensures that the robber cannot be captured in round 1 since $\mathcal{R}_0$ is connected to neither $\mathcal{S}_0$ nor $\mathcal{S}_1$. Moreover, we observe that both $\mathcal{S}_0$ and $\mathcal{S}_1$ are connected neither to any node in $\mathcal{A}$, nor to $\mathcal{R}_{-1}$, $\mathcal{R}_0$ or $\mathcal{R}_1$, and that no node from $\mathcal{T}$ is connected to $\mathcal{R}_0$. Thus, looking at Table 2, we can conclude that there is at most[3] one case in which the robber can be captured at time $t+2$, namely, that at time t she is in some node $\mathcal{R}_i$ and the cop combination $\{c_0(t+1), c_1(t+1)\}$ covers all exits of $\mathcal{R}_i$ as well as $\mathcal{R}_{i-1}$, $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$. Then she will stay in $\mathcal{R}_i$ and can be captured at time $t+2$.

[3] That the robber can actually be captured will be shown in Section 3.7.

By Lemma 9, there are two possibilities how the cops can cover those exits: Either they are in some exit-covering cop combination or one of the cops is in some node $\mathcal{R}_j$ with $j \equiv i \pmod 3$. By Lemma 11, the latter case cannot occur.

Thus, the only case where the robber can possibly be captured at time $t+2$ is that she is in some node $\mathcal{R}_i$ at time t and the cops are in some exit-covering cop combination at time $t+1$ upon which the robber stays in $\mathcal{R}_i$ and can be captured there by the cops at time $t+2$. In order to capture the robber in its $\mathcal{R}$ node at time $t+2$, one cop has to cover this $\mathcal{R}$ node at time $t+1$. By Lemma 9 and Table 1, we obtain that $\{c_0(t+1), c_1(t+1)\} = \{\mathcal{C}_q, \mathcal{D}_{q'}\}$ for some q, q'. Since any node from $\mathcal{C}$ covers at most 2 of any 3 consecutively indexed nodes from $\mathcal{R}$ and any node from $\mathcal{D}$ does not cover any node from $\mathcal{R}$, this implies that $\{c_0(t+1), c_1(t+1)\}$ covers at most 2 of any 3 consecutively indexed nodes from $\mathcal{R}$. Thus, the robber can only be captured at time $t+2$ if she is in $\mathcal{R}_{\hat{n}}$ or $\mathcal{R}_{-\hat{n}}$ at time $t+1$; otherwise she would go to an uncovered node from $\{\mathcal{R}_{i-1}, \mathcal{R}_i, \mathcal{R}_{i+1}\}$ according to Table 2. If the robber is indeed in $\mathcal{R}_{\hat{n}}$ or $\mathcal{R}_{-\hat{n}}$ at time $t+1$, she will stay in this node if her current node and the neighboring $\mathcal{R}$ node are covered (which is the only case in which she will be captured at time $t+2$). We conclude that the robber can only be captured in $\mathcal{R}_{\hat{n}}$ or $\mathcal{R}_{-\hat{n}}$. ◀

Proof of Lemma 13. We start by observing that for any $2 \leq i \leq \hat{n}$, the only case where the robber may move to $\mathcal{R}_i$ is if she is in some node in $\{\mathcal{R}_{i-1}, \mathcal{R}_i, \mathcal{R}_{i+1}\}$ whose exits are all covered. This implies that the *first* time the robber moves to $\mathcal{R}_i$ she comes from $\mathcal{R}_{i-1}$ (since the path from $\mathcal{R}_2$ to $\mathcal{R}_{\hat{n}}$ can only be entered via $\mathcal{R}_1$ by the robber). Also, the *last* time the robber moves to $\mathcal{R}_i$ from some node $\neq \mathcal{R}_i$ *before* she moves to $\mathcal{R}_{i+1}$ for the *first* time, she has to come from $\mathcal{R}_{i-1}$.

Now, let $3 \leq i \leq \hat{n}$ and denote by t the last time that the robber enters $\mathcal{R}_{i-1}$ before she moves to $\mathcal{R}_i$ for the first time and by t' the first time that the robber enters $\mathcal{R}_i$. Assume for the moment that the robber actually goes to those two $\mathcal{R}$ nodes at some point. By the above observations, we have that $r(t-1) = \mathcal{R}_{i-2}$, $r(t) = r(t+1) = \cdots = r(t'-1) = \mathcal{R}_{i-1}$ and $r(t') = \mathcal{R}_i$.

Recall the definition of a forcing proper exit-blocking cop combination. Note that a proper exit-blocking cop combination that covers two adjacent $\mathcal{R}$ nodes is always forcing, by Table 1. Checking Table 2, we see that the robber only (possibly) moves from $\mathcal{R}_{i-2}$ to $\mathcal{R}_{i-1}$ if $\mathcal{R}_{i-2}$, $\mathcal{R}_{i-3}$ and all exits of $\mathcal{R}_{i-2}$ are covered. Thus, by Lemma 9 and Lemma 11, $\{c_0(t), c_1(t)\}$ is a forcing proper exit-covering cop combination (proper because nodes from $\mathcal{S}$ or $\mathcal{T}$ do not cover any nodes from $\mathcal{R}$). Analogously, we obtain that $\{c_0(t'), c_1(t')\}$ is a forcing proper exit-covering cop combination. By Lemma 9, we can write those cop combinations as $\{\mathcal{C}_j, \mathcal{D}_h\} = \{c_0(t), c_1(t)\}$ and $\{\mathcal{C}_{j'}, \mathcal{D}_{h'}\} = \{c_0(t'), c_1(t')\}$, for some j, h, j', h'.

Note that any forcing proper exit-covering cop combination can cover at most 2 of any 3 consecutively indexed nodes from $\mathcal{R}$, as can be seen from Table 1. Since, as observed above, $\{\mathcal{C}_j, \mathcal{D}_h\} = \{c_0(t), c_1(t)\}$ has to cover $\mathcal{R}_{i-2}$ and $\mathcal{R}_{i-3}$, and, similarly, $\{\mathcal{C}_{j'}, \mathcal{D}_{h'}\} = \{c_0(t'), c_1(t')\}$ has to cover $\mathcal{R}_{i-1}$ and $\mathcal{R}_{i-2}$, it follows that $j \neq j'$.

Since the robber stays in $\mathcal{R}_{i-1}$ from time t to time $t'-1$, the cops must have covered $\mathcal{R}_{i-2}$ and all exits of $\mathcal{R}_{i-1}$ during that time frame. In other words, $\{c_0(t+1), c_1(t+1)\}, \{c_0(t+2), c_1(t+2)\}, \ldots, \{c_0(t'-1), c_1(t'-1)\}$ are proper exit-blocking cop combinations. Now, we have collected all ingredients to actually apply Lemma 10. We obtain $t'-t \geq \hat{n}/3 \cdot (\hat{n}/3 - 1) \in \Omega(\hat{n}^2)$. This implies that for any $3 \leq i \leq \hat{n}$, the time between the first visit of $\mathcal{R}_{i-1}$ by the robber and the first visit of $\mathcal{R}_i$ is in $\Omega(\hat{n}^2)$.

By Lemma 12, the robber can only be captured in $\mathcal{R}_{\hat{n}}$ or $\mathcal{R}_{-\hat{n}}$. Assume that she will be captured in $\mathcal{R}_{\hat{n}}$. As there are $\Omega(\hat{n})$ nodes along the $\mathcal{R}$ path between $\mathcal{R}_2$ and $\mathcal{R}_{\hat{n}}$ and forcing the robber to go from one node of the path to the next one takes the cops time $\Omega(\hat{n}^2)$, we obtain a lower bound of $\Omega(\hat{n}^3)$. Analogously, we obtain a lower bound of $\Omega(\hat{n}^3)$ if we assume that the robber will be captured in $\mathcal{R}_{-\hat{n}}$. ◀

Proof of Lemma 14. If $j \notin \{\hat{n}/3-1, 2\hat{n}/3-1, \hat{n}-1\}$, then $\mathcal{D}_{j+1 \pmod{\hat{n}}}$ covers exactly the same nodes from $\mathcal{X}$ as $\mathcal{D}_j$. Thus, in that case, $(\mathcal{C}_i, \mathcal{D}_{j+1 \pmod{\hat{n}}})$ must be exit-blocking if $(\mathcal{C}_i, \mathcal{D}_j)$ is exit-blocking.

If $j \in \{\hat{n}/3-1, 2\hat{n}/3-1, \hat{n}-1\}$, then it is straightforward to check with Table 1 that the nodes in $\mathcal{X}$ that $\mathcal{D}_{j+1 \pmod{\hat{n}}}$ does not cover, but $\mathcal{D}_j$ covers, are exactly the nodes in $\mathcal{X}$ that $\mathcal{C}_{i+1 \pmod{\hat{n}}}$ covers, but $\mathcal{C}_i$ does not, and vice versa. Thus, in that case, $(\mathcal{C}_{i+1 \pmod{\hat{n}}}, \mathcal{D}_{j+1 \pmod{\hat{n}}})$ covers exactly the same exits as $(\mathcal{C}_i, \mathcal{D}_j)$. ◀

Proof of Lemma 15. By Lemma 14 and Table 3, the cops always stay in proper exit-blocking cop combinations from time $t+1$ onwards, as long as the robber does not leave $\mathcal{R}$. Since any proper exit-blocking cop combination not only covers all exits of all $\mathcal{R}$ nodes, but also all $\mathcal{A}$ nodes and $\mathcal{P}$ (since the cop combination contains a $\mathcal{D}$ node), the only node outside of $\mathcal{R}$ where the robber could go without being captured immediately is some node $\mathcal{C}_i$. So suppose $r(t') = \mathcal{C}_i$ for some i. Then, according to Table 3, the cops move to $\mathcal{S}_0$ and $\mathcal{P}$ at time $t'+1$. Now, checking Table 1, we see that any neighbor of $\mathcal{C}_i$, and also $\mathcal{C}_i$ itself, is covered by $\mathcal{S}_0$ or $\mathcal{P}$. Thus, the robber will be captured at time $t'+2$. ◀

Proof of Lemma 16. As specified in Section 3.5, the cops start in $c_0(0) = \mathcal{S}_0$ and $c_1(0) = \mathcal{S}_1$. If the robber starts in a node in $\mathcal{U} \cup \mathcal{S} \cup \mathcal{T} \cup \mathcal{C} \cup \mathcal{X}$, then, by Table 1, she will be caught at time 1, so suppose she starts in a different node, i.e., in a node in $\mathcal{A} \cup \mathcal{R} \cup \mathcal{D} \cup \{\mathcal{P}\}$.

If $r(0) = \mathcal{A}_i$ for some i, then, according to Table 3, the cops go to $c_0(1) = \mathcal{S}_0$ and $c_1(1) = \mathcal{T}_i$, a cop combination that covers all neighbors of $\mathcal{A}_i$ except $\mathcal{R}_0$. Thus, in order to avoid being captured at time 2, the robber has to move to $r(1) = \mathcal{R}_0$.

If $r(0) \notin \mathcal{A}$, i.e., $\mathcal{R} \cup \mathcal{D} \cup \{\mathcal{P}\}$, then the cops go to $c_0(1) = \mathcal{S}_0$ and $c_1(1) = \mathcal{T}_i$ for some arbitrary i. This cop combination covers all nodes in $\mathcal{S} \cup \mathcal{A} \cup \mathcal{T} \cup \mathcal{C} \cup \mathcal{D} \cup \{\mathcal{P}\} \cup \mathcal{X}$ and since $r(0)$ is not connected to any node in $\mathcal{U}$, the only nodes where the robber can go at time 1 in order to avoid capture at time 2 are nodes in $\mathcal{R}$.

Therefore, as in both cases the robber can only avoid immediate capture by going to an $\mathcal{R}$ node, suppose that $r(1) \in \mathcal{R}$. At time 2, according to Table 3, the cops move from $(\mathcal{S}_0, \mathcal{T}_i)$ to $(c_0(2), c_1(2)) = (\mathcal{C}_0, \mathcal{D}_0)$.

Now, by Lemma 15, we can assume for the remainder of the proof that the robber does not leave $\mathcal{R}$ anymore. Observe that the subgraph of our graph induced by $\mathcal{R}$ is simply a path. Since the robber stays in $\mathcal{R}$, the cops will always perform the "same" step according to Table 3, namely, they go from some cop combination $(\mathcal{C}_i, \mathcal{D}_j)$ to $(\mathcal{C}_{i+1 \pmod{\hat{n}}}, \mathcal{D}_{j+1 \pmod{\hat{n}}})$ or $(\mathcal{C}_i, \mathcal{D}_{j+1 \pmod{\hat{n}}})$.

How long can Cop 0 possibly stay in $\mathcal{C}_i$ while performing these steps? After at most $\hat{n}/3-1$ steps Cop 1 reaches node $\mathcal{D}_{\hat{n}/3-1}$, $\mathcal{D}_{2\hat{n}/3-1}$ or $\mathcal{D}_{\hat{n}-1}$. Then, in the next step, Cop 1 goes to a $\mathcal{D}$ node that covers a different set of $\mathcal{X}$ nodes. Since, for each $0 \le h \le 2$, each $\mathcal{C}$ node and each $\mathcal{D}$ node covers exactly 3 nodes from $\mathcal{X}^h$ and $\mathcal{X}^h$ contains 6 nodes, the $\mathcal{C}$ node and the $\mathcal{D}$ node in a proper exit-blocking cop combination must cover complementary sets of $\mathcal{X}$ nodes. Thus, if Cop 1 changes the set of nodes in $\mathcal{X}$ she covers, then Cop 0 is required to move. Hence after at most $\hat{n}/3$ steps, Cop 0 moves from $\mathcal{C}_i$ to $\mathcal{C}_{i+1 \pmod{\hat{n}}}$. Note that Lemma 14 ensures that the cops indeed always cover all exits.

Using Table 3, the above discussion shows that the movement of Cop 0 looks as follows (starting from $c_0(2) = \mathcal{C}_0$): She stays in some node $\mathcal{C}_i$ for at most $\hat{n}/3$ steps, then she moves to the next $\mathcal{C}$ node (modulo $\hat{n}$) in which she again stays for at most $\hat{n}/3$ steps, and so on.

Since $-\hat{n} \equiv 0 \pmod 3$, Cop 0 covers $\mathcal{R}_{-\hat{n}}$ at time 2, according to Table 1. Thus, at time 2, the robber has to be in some node $\mathcal{R}_q$ with $q > -\hat{n}$ (in order to not be captured immediately). During moving in $\mathcal{C}$, Cop 0 keeps $\mathcal{R}_{-\hat{n}}$ covered until she reaches $\mathcal{C}_{\hat{n}/3}$. When she is in $\mathcal{C}_{\hat{n}/3}$ she not only covers $\mathcal{R}_{-\hat{n}}$, but also the next $\mathcal{R}$ node, $\mathcal{R}_{-\hat{n}+1}$. Thus, now the robber (who previously stayed in some node $\mathcal{R}_q$ with $q > -\hat{n}$) has to be in some node $\mathcal{R}_{q'}$ with $q' > -\hat{n} + 1$.

By inductively applying this argument, the set of $\mathcal{R}$ nodes the robber can be in without being captured immediately, shrinks by one node each time Cop 0 reaches a node from $\{\mathcal{C}_0, \mathcal{C}_{\hat{n}/3}, \mathcal{C}_{2\hat{n}/3}\}$. When there are no more $\mathcal{R}$ nodes left in that set, the robber will be captured immediately.[4] Note that because of the path topology of the subgraph induced by $\mathcal{R}$, in order to shrink the node set the robber can stay in, it is sufficient for Cop 0 to cover one node from the path and (slowly) move it to the side the robber is in while making sure that when moving from covering one node to covering the next, there is a point in time where both nodes are covered.[5]

Collecting all the information from the above discussion, we obtain the following picture: Moving from one $\mathcal{C}$ node to the next $\mathcal{C}$ node takes Cop 0 at most $\hat{n}/3 \in \mathcal{O}(\hat{n})$ time steps. Whenever Cop 0 reaches a node from $\{\mathcal{C}_0, \mathcal{C}_{\hat{n}/3}, \mathcal{C}_{2\hat{n}/3}\}$ (which happens at least once in any consecutive $\hat{n}/3 \cdot \hat{n}/3 \in \mathcal{O}(\hat{n}^2)$ time steps) the number of $\mathcal{R}$ nodes the robber can be in without being captured immediately decreases by 1. As there are only $\mathcal{O}(\hat{n})$ $\mathcal{R}$ nodes in total, the robber will be captured in time $\mathcal{O}(\hat{n}^3)$. ◀

B The General Case

In this section, we will generalize our lower bound construction to arbitrary numbers of cops up to cop numbers in $\Theta(\sqrt{n})$. This upper bound up to which our lower bound construction works coincides with the conjectured minimum number of cops with which the robber can be captured in all graphs.

In the following, we give a high-level overview of how we change $\{G_{\hat{n}}\}_{\hat{n} \geq 3, \hat{n} \equiv 0 \pmod 3}$ in order to obtain a family $\{G^k_{\hat{n}}\}_{\hat{n} \geq k, \hat{n} \equiv 0 \pmod 3}$ of graphs that will yield a lower bound of $(\Omega(\hat{n}))^{k+1}$ for the capture time of k cops in k-cop-win graphs. The main idea of the construction, as explained in Section 3, remains the same. In the 2-cop case, the cops started in $\mathcal{S}_0$ and $\mathcal{S}_1$ in order to flush the robber out of her preferred component of the graph and into the $\mathcal{R}$ path, and then went to the $\mathcal{C}$ and $\mathcal{D}$ cycles and circled through the nodes there again and again in order to force the robber to one end of her $\mathcal{R}$ path.

For the general k-cop case, we add $k-2$ special nodes $\mathcal{S}_2, \ldots, \mathcal{S}_{k-1}$ and $k-2$ new node sets $\mathcal{D}^1, \ldots, \mathcal{D}^{k-2}$ for the additional $k-2$ cops (and rename $\mathcal{D}$ by $\mathcal{D}^0$). Now, the robber can only be flushed out of her preferred graph component $\mathcal{U}$ if the k cops go to the cop combination $\{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$. In that case, the robber can be forced to go to the $\mathcal{R}$ path and then Cop 0 goes to $\mathcal{C}$ while, for any $i \geq 1$, Cop i goes to $\mathcal{D}^{i-1}$. Like in $\mathcal{D}^0$, the nodes in any $\mathcal{D}^{j\geq 1}$ are connected by edges in a circular fashion and each $\mathcal{D}^{j\geq 1}$ contains $\hat{n}$ nodes.

Again, we want the cops to have to go through a certain combined movement in order to force the robber to move one step on her path, and we want this combined movement to

[4] This holds under the assumption we made that the robber stays in $\mathcal{R}$. If she does not stay in $\mathcal{R}$ she can increase the capture time by 1 by moving to some $\mathcal{C}$ node.

[5] Indeed, Cop 0 could also start by covering node $\mathcal{R}_0$ and then move the covered node to the side of the path the robber is in. This accelerates the capture time roughly by a factor of 2. The downside is that it makes the cops' strategy a bit more complex to describe which is why we chose the presented approach.

take a long time (in general $(\Omega(\hat{n}))^k$). Similarly to the 2-cop case, the cops cannot simply advance to their desired forcing cop combination by moving there without coordinating, but they have to constantly guard the exits of the $\mathcal{R}$ node the robber occupies.

In the 2-cop case, between taking two steps on her $\mathcal{C}$ cycle, Cop 0 had to wait for Cop 1 to travel a third of her $\mathcal{D}$ cycle because Cop 1 had to reach a node where she covers the exits Cop 0 would not cover anymore when going to the new $\mathcal{C}$ node. For the k-cop case, we generalize this idea by doing the following: For any surplus Cop i compared to the 2-cop case, we add 6 exits to any $\mathcal{R}$ node[6]. These 6 exits can only be covered by cops $i-1$ and i. Moreover, analogous to the construction for the "initial" 6 exits which only cops 0 and 1 can cover, we ensure that between any two steps of Cop i (in the same direction, clockwise or counterclockwise), Cop $i-1$ has to travel a third of her $\mathcal{D}^{i-2}$ cycle in order to always keep all of the 6 exits, that the combination of Cop $i-1$ and Cop i is responsible for, covered.

Thus, Cop $k-1$ has to take $\Omega(\hat{n})$ steps between any two consecutive steps of Cop $k-2$, Cop $k-2$ has to take $\Omega(\hat{n})$ steps between any two consecutive steps of Cop $k-3$, and so on, yielding a lower bound of $(\Omega(\hat{n}))^k$ for the cops to go from a cop combination that forces the robber to take a step on her $\mathcal{R}$ path to the next such combination. Note that Cop 0, i.e., the cop that actually forces the robber to move, has to take $\Omega(\hat{n})$ steps in order to get from one forcing cop combination to the next one.

Since the robber reaches the end of her path in $\Omega(\hat{n})$ steps, we obtain a capture time of $(\Omega(\hat{n}))^{k+1}$.

While we can choose any multiple of 3 as the number $\hat{n}$ determining the number of nodes in the $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{D}^i$ components of the graph, the size of most of the other components of the graph depends on the number of cops. More specifically, for technical reasons, the size of $\mathcal{U}$ depends quadratically[7] on k whereas the sizes of $\mathcal{S}$, $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{X}$ depend linearly on k. Thus, we obtain that $n \in \Theta(k^2 + k\hat{n})$, and for our lower bound construction to work, we have to require that $k \in \mathcal{O}(\sqrt{n})$.[8] Moreover, for a constant k, $\hat{n}$ is linear in the number n of nodes, yielding a tight lower bound of $\Omega(n^{k+1})$ for the capture time. If k is not constant, then $\hat{n}$ is in $\Theta(n/k)$ which yields a lower bound of $(\Omega(n/k))^{k+1}$. In particular, for $k \in \Theta(\sqrt{n})$, we obtain a lower bound of $(\Omega(\sqrt{n}))^{\Theta(\sqrt{n})}$. Thus, surprisingly, there are graphs in which the minimum number of cops sufficient to capture the robber actually needs time exponential in $\sqrt{n}$ to capture the robber.

B.1 The Graph Construction

Recall the construction of the preliminary graph $G_{\mathcal{E},\mathcal{L}}$ from which we borrowed nodes in order to obtain the components $\mathcal{E}$, $\mathcal{L}^*$ and $\mathcal{X}$ of $G_{\hat{n}}$. We want to adapt the construction to the case of k cops such that Lemmas 2, 3, and 4, also adapted to the k-cop case, still hold.

In order to achieve this, we simply change the sizes of the sets $\mathcal{E}$ and $\mathcal{L}$ used in the construction of $G_{\mathcal{E},\mathcal{L}}$. Let z be the smallest integer ≥ 10 such that $2k+z$ is a prime. For reasons that will become clear soon, we replace every "9", "10" and "11" in the construction of $G_{\mathcal{E},\mathcal{L}}$, by $2k+z-2$, $2k+z-1$ and $2k+z$, respectively. Thus,

$$\mathcal{E} = \{\mathcal{E}_{i,j} \mid 0 \leq i \leq 2k+z-1 \wedge 0 \leq j \leq 2k+z-1\}$$

[6] Note that $\mathcal{R}_j$ and $\mathcal{R}_h$ share the exits if $j \equiv h \pmod 3$.
[7] To be precise, we need Bertrand's postulate to obtain this result, since the size of $\mathcal{U}$ is the square of the first prime that follows $k+10$.
[8] To be more precise, there is some universal constant $\beta \geq 1$, so that k is bounded from above by $\sqrt{n}/\beta$.

and

$$\mathcal{L} = \{\mathcal{L}_{i,j} \mid 0 \le i \le 2k+z-2 \wedge 0 \le j \le 2k+z-1\} ,$$

where $\mathcal{L}_{i,j} = \{\mathcal{E}_{h,h(i+1)+j \pmod{2k+z}} \mid 0 \le h \le 2k+z-1\}$. Again, we consider the associated bipartite incidence graph which we denote by $G^k_{\mathcal{E},\mathcal{L}}$.[9]

In the following, we give analogous versions of Lemmas 2, 3, and 4 for $G^k_{\mathcal{E},\mathcal{L}}$. Since in the proofs of the original lemmas, regarding the specific size of $G_{\mathcal{E},\mathcal{L}}$ we use only the property that 11 is prime, we can use analogous proofs to show our new lemmas.

▶ **Lemma 17.** *In $G^k_{\mathcal{E},\mathcal{L}}$, any two nodes in $\mathcal{E}$ have at most one common neighbor in $\mathcal{L}$. Furthermore, any two nodes in $\mathcal{L}$ have at most one common neighbor in $\mathcal{E}$.*

▶ **Lemma 18.** *Let $i \in \{0, \dots, 2k+z-1\}$ be fixed. Any node from $\mathcal{L}$ has exactly one neighbor in $G^k_{\mathcal{E},\mathcal{L}}$ of the form $\mathcal{E}_{i,j}$ and exactly one neighbor in $G^k_{\mathcal{E},\mathcal{L}}$ of the form $\mathcal{E}_{j,i}$.*

▶ **Lemma 19.** *Let $i \in \{0, \dots, 2k+z-2\}$ be fixed. Any node from $\mathcal{E}$ has exactly one neighbor in $G^k_{\mathcal{E},\mathcal{L}}$ of the form $\mathcal{L}_{i,j}$.*

In general, we will construct $G^k_{\hat{n}}$ analogously to the construction of $G_{\hat{n}}$. Again, we will *borrow* nodes, this time from $G^k_{\mathcal{E},\mathcal{L}}$. Moreover, we will add new node sets $\mathcal{D}^1, \dots, \mathcal{D}^{k-2}$, consisting of $\hat{n}$ nodes each, and increase the sizes of the components from $G_{\hat{n}}$ as follows:

$$V(G^k_{\hat{n}}) = \mathcal{E} \cup \mathcal{L}^* \cup \mathcal{S} \cup \mathcal{A} \cup \mathcal{T} \cup \mathcal{R} \cup \mathcal{C} \cup \mathcal{D}^0 \cup \mathcal{D}^1 \cup \dots \cup \mathcal{D}^{k-2} \cup \{\mathcal{P}\} \cup \mathcal{X}$$

where

$$\begin{aligned}
\mathcal{L}^* &= \{\mathcal{L}_{i,j} \mid 0 \le i \le 2k-1 \wedge 0 \le j \le 2k+z-1\} ,\\
\mathcal{S} &= \{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\} ,\\
\mathcal{A} &= \{\mathcal{A}_0, \dots, \mathcal{A}_{2k-1}\} ,\\
\mathcal{T} &= \{\mathcal{T}_0, \dots, \mathcal{T}_{2k-1}\} ,\\
\mathcal{R} &= \{\mathcal{R}_{-\hat{n}}, \dots, \mathcal{R}_{\hat{n}}\} ,\\
\mathcal{C} &= \{\mathcal{C}_0, \dots, \mathcal{C}_{\hat{n}-1}\} ,\\
\mathcal{D}^j &= \{\mathcal{D}^j_0, \dots, \mathcal{D}^j_{\hat{n}-1}\} \text{ for all } 0 \le j \le k-2 ,\\
\mathcal{X} &= \mathcal{X}^0 \cup \mathcal{X}^1 \cup \mathcal{X}^2 \text{ and}\\
\mathcal{X}^j &= \{\mathcal{X}^j_0, \dots, \mathcal{X}^j_{6(k-1)-1}\} \text{ for all } 0 \le j \le 2 .
\end{aligned}$$

The edges of $G^k_{\hat{n}}$ are specified in Tables 4, 5, and 6. Again, we set $\mathcal{U} = \mathcal{E} \cup \mathcal{L}^*$ and call the $\mathcal{X}$ nodes (connected to some node $\mathcal{R}_i$) *exits (of $\mathcal{R}_i$)*.

As in the 2-cop case, $\mathcal{E}$, $\mathcal{L}^*$ and $\mathcal{X}$ are borrowed from $G^k_{\mathcal{E},\mathcal{L}}$ where $\mathcal{X}$ is considered as a subset of $\mathcal{L} \setminus \mathcal{L}^*$. To ensure that no node in $\mathcal{E}$ covers too many exits of some $\mathcal{R}$ node, the exits of any $\mathcal{R}$ node are borrowed (disjointly) from $\mathcal{L} \setminus \mathcal{L}^*$ nodes of at most 3 different slopes. More precisely,

$$\begin{aligned}
&\mathcal{X}^0 \text{ is borrowed from } \{\mathcal{L}_{h,j} \mid 2k \le h \le 2k+2 \wedge 0 \le j \le 2k+z-1\} ,\\
&\mathcal{X}^1 \text{ is borrowed from } \{\mathcal{L}_{h,j} \mid 2k+3 \le h \le 2k+5 \wedge 0 \le j \le 2k+z-1\} ,\\
&\mathcal{X}^2 \text{ is borrowed from } \{\mathcal{L}_{h,j} \mid 2k+6 \le h \le 2k+8 \wedge 0 \le j \le 2k+z-1\} ,
\end{aligned}$$

[9] We may consider $G_{\mathcal{E},\mathcal{L}}$ as a version of $G^2_{\mathcal{E},\mathcal{L}}$, where we simplified the construction by choosing $z = 7$ (instead of $z = 13$).

$\mathcal{E}_{i,j}$	$\mathcal{L}^*$ nodes as determined by $G^k_{\mathcal{E},\mathcal{L}}$ $\mathcal{S}_{i \pmod k}$ $\mathcal{A}_{j \pmod{2k}}$ $\mathcal{T}_{j \pmod{2k}}$ if $i \pmod k = 1$ $\mathcal{X}$ nodes as determined by $G^k_{\mathcal{E},\mathcal{L}}$
$\mathcal{L}_{i,j}$	$\mathcal{E}$ nodes as determined by $G^k_{\mathcal{E},\mathcal{L}}$ $\mathcal{S}_{i \pmod k}$ $\mathcal{A}_{i \pmod{2k}}$ $\mathcal{T}_{i \pmod{2k}}$

Table 4 Edges in $G^k_{\hat{n}}$.

$\mathcal{S}_0$	$\mathcal{C}_i$ for all i $\mathcal{X}^j_i$, where $j \in \{0,1,2\}$ and $i \in \{0,1,2\}$ every $\mathcal{E}_{i,j}$ with $i \pmod k = 0$ every $\mathcal{L}_{i,j}$ with $i \pmod k = 0$
$\mathcal{S}_1$	$\mathcal{T}_i$ for all i $\mathcal{X}^j_i$, where $j \in \{0,1,2\}$ and $i \in \{3,4,5\}$ $\mathcal{X}^j_i$, where $j \in \{0,1,2\}$ and $i \in \{6,7,8\}$ if $k \geq 3$ every $\mathcal{E}_{i,j}$ with $i \pmod k = 1$ every $\mathcal{L}_{i,j}$ with $i \pmod k = 1$
$\mathcal{S}_{i\geq 2}$	D^i_j for all j $\mathcal{X}^j_h$, where $j \in \{0,1,2\}$ and $h \in \{6(i-1)+3, 6(i-1)+4, 6(i-1)+5\}$ $\mathcal{X}^j_h$, where $j \in \{0,1,2\}$ and $h \in \{6i, 6i+1, 6i+2\}$ for all $i \leq k-2$ every $\mathcal{E}_{h,j}$ with $h \pmod k = i$ every $\mathcal{L}_{h,j}$ with $h \pmod k = i$
$\mathcal{A}_i$	$\mathcal{T}_j$ for all j $\mathcal{R}_0$ $\mathcal{D}^0_j$ for all j $\mathcal{E}_{j,h}$ for all $h \pmod{2k} = i$ $\mathcal{L}_{j,h}$ for all $j \pmod{2k} = i$
$\mathcal{T}_i$	$\mathcal{S}_1$ $\mathcal{A}_j$ for all j $\mathcal{T}_j$ for all $j \neq i$ $\mathcal{D}^0_j$ for all j $\mathcal{P}$ $\mathcal{X}^j_h$, where $j \in \{0,1,2\}$ and $h \in \{3,4,5\}$ $\mathcal{X}^j_h$, where $j \in \{0,1,2\}$ and $h \in \{6,7,8\}$ if $k \geq 3$ $\mathcal{E}_{j,h}$ for all $j \pmod k = 1$ and $h \pmod{2k} = i$ $\mathcal{L}_{j,h}$ for all $j \pmod{2k} = i$

Table 5 Edges in $G^k_{\hat{n}}$.

$\mathcal{R}_i$	$\mathcal{A}_j$ for all j if $i = 0$ $\mathcal{R}_j$ for $j = i - 1$ and $j = i + 1$ $\mathcal{C}_j$ for all $0 \leq j \leq \hat{n}/3$ if $i \pmod 3 = 0$ $\mathcal{C}_j$ for all $\hat{n}/3 \leq j \leq 2\hat{n}/3$ if $i \pmod 3 = 1$ $\mathcal{C}_0$ and $\mathcal{C}_j$ for all $2\hat{n}/3 \leq j \leq \hat{n} - 1$ if $i \pmod 3 = 2$ $\mathcal{P}$ $\mathcal{X}_h^j$ where $j = i \pmod 3$ and $h \in \{0, \ldots, 6(k-1) - 1\}$
$\mathcal{C}_i$	$\mathcal{S}_0$ $\mathcal{R}_j$ for all $j \pmod 3 = 0$ if $0 \leq i \leq \hat{n}/3$ $\mathcal{R}_j$ for all $j \pmod 3 = 1$ if $\hat{n}/3 \leq i \leq 2\hat{n}/3$ $\mathcal{R}_j$ for all $j \pmod 3 = 2$ if $i = 0$ or $2\hat{n}/3 \leq i \leq \hat{n} - 1$ $\mathcal{C}_j$ for $j \equiv i - 1 \pmod{\hat{n}}$ and $j \equiv i + 1 \pmod{\hat{n}}$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{0,1,3\}$ if $i \pmod 3 = 0$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{0,2,3\}$ if $i \pmod 3 = 1$ $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{1,2,3\}$ if $i \pmod 3 = 2$

$\mathcal{D}_i^0$	$\mathcal{A}_j$ for all j
	$\mathcal{T}_j$ for all j
	$\mathcal{D}_j^0$ for $j \equiv i - 1 \pmod{\hat{n}}$ and $j \equiv i + 1 \pmod{\hat{n}}$
	$\mathcal{P}$
	$\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{2,4,5\}$ if $0 \leq i \leq \hat{n}/3 - 1$
	$\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{1,4,5\}$ if $\hat{n}/3 \leq i \leq 2\hat{n}/3 - 1$
	$\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{0,4,5\}$ if $2\hat{n}/3 \leq i \leq \hat{n} - 1$
	additionally, if $k \geq 3$:
	    $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{6,7,9\}$ if $i \pmod 3 = 0$
	    $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{6,8,9\}$ if $i \pmod 3 = 1$
	    $\mathcal{X}_h^j$ where $j \in \{0,1,2\}$ and $h \in \{7,8,9\}$ if $i \pmod 3 = 2$

$\mathcal{D}_i^{j\geq 1}$	$\mathcal{S}_{j+1}$
	$\mathcal{D}_h^j$ for $h \equiv i - 1 \pmod{\hat{n}}$ and $h \equiv i + 1 \pmod{\hat{n}}$
	$\mathcal{X}_h^q$ where $q \in \{0,1,2\}$ and $h \in \{6j+2, 6j+4, 6j+5\}$ if $0 \leq i \leq \hat{n}/3 - 1$
	$\mathcal{X}_h^q$ where $q \in \{0,1,2\}$ and $h \in \{6j+1, 6j+4, 6j+5\}$ if $\hat{n}/3 \leq i \leq 2\hat{n}/3 - 1$
	$\mathcal{X}_h^q$ where $q \in \{0,1,2\}$ and $h \in \{6j+0, 6j+4, 6j+5\}$ if $2\hat{n}/3 \leq i \leq \hat{n} - 1$
	additionally, if $j \leq k - 3$:
	    $\mathcal{X}_h^q$ where $q \in \{0,1,2\}$ and $h \in \{6(j+1)+0, 6(j+1)+1, 6(j+1)+3\}$ if $i \pmod 3 = 0$
	    $\mathcal{X}_h^q$ where $q \in \{0,1,2\}$ and $h \in \{6(j+1)+0, 6(j+1)+2, 6(j+1)+3\}$ if $i \pmod 3 = 1$
	    $\mathcal{X}_h^q$ where $q \in \{0,1,2\}$ and $h \in \{6(j+1)+1, 6(j+1)+2, 6(j+1)+3\}$ if $i \pmod 3 = 2$

$\mathcal{P}$	$\mathcal{T}_i$ for all i $\mathcal{R}_i$ for all i $\mathcal{D}_i^0$ for all i $\mathcal{X}_i^j$ where $j \in \{0,1,2\}$ and $i = 3$

Table 6 Edges in $G_{\hat{n}}^k$.

where, we specify in particular that

$$\mathcal{X}_i^0 \text{ is borrowed from } \{\mathcal{L}_{2k,j} \mid 0 \le j \le 2k+z-1\} ,$$
$$\mathcal{X}_i^1 \text{ is borrowed from } \{\mathcal{L}_{2k+3,j} \mid 0 \le j \le 2k+z-1\} ,$$
$$\mathcal{X}_i^2 \text{ is borrowed from } \{\mathcal{L}_{2k+6,j} \mid 0 \le j \le 2k+z-1\} ,$$

if $0 \le i \le 5 \vee 6(k-2) \le i \le 6(k-1)-1 \vee i \equiv 5 \pmod 6$.

As long as the above conditions are met, we do not care about the explicit choice of $\mathcal{X}$ as a subset of $\mathcal{L} \setminus \mathcal{L}^*$. Note that the (perhaps a bit confusing) above specification of some $\mathcal{X}$ nodes is only relevant for Lemma 24. Furthermore, we see the reason for the exact definition of z: To have nodes of the form $\mathcal{L}_{2k+8,j}$ available as required above, we need that $z \ge 10$ (i.e., $2k+z-2 \ge 2k+8$). It is straightforward to verify that if $z \ge 10$, there are enough nodes in the node sets we (disjointly) borrow from in the above specifications.

We obtain an analogous version of Corollary 5 from Lemma 17.

▶ **Corollary 20.** *In $G_{\hat{n}}^k$, any two nodes in $\mathcal{E}$ have at most one common neighbor in $\mathcal{L}^*$. Moreover, any two nodes in $\mathcal{L}^* \cup \mathcal{X}$ have at most one common neighbor in $\mathcal{E}$.*

B.2 Observations

We show in the following that analogous versions of Lemmas 6, 7, 8, 9 and 10 hold also in the k-cop case. Since the proofs of the original lemmas were already formulated with regard to the generalization, almost all of the arguments can be transferred easily to the k-cop case. Thus, for convenience, we will only point out the differences in the proofs of the original and the k-cop case versions.

▶ **Lemma 21.** *Consider any $u \in \mathcal{U}$. The only cop combination not containing u that covers all neighbors of u in $\mathcal{U}$ is $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$.*

Proof. Analogously to the proof of Lemma 6, if $u \in \mathcal{E}$, then the nodes from $\mathcal{S}$ are the only nodes that cover more than one of u's $2k$ neighbors in $\mathcal{U}$. Since any $\mathcal{S}$ covers exactly 2 of those neighbors, the cops have to be in $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$ in order to cover all of those neighbors.

If $u \in \mathcal{L}^*$, then analogously any node from $\mathcal{S}$ covers strictly more of u's neighbors in $\mathcal{U}$ than any node $\notin \mathcal{S} \cup \{u\}$. Since no node in $\mathcal{U}$ is covered by more than one node from $\mathcal{S}$, any cop combination different from $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$ covers strictly less than all of u's neighbors in $\mathcal{U}$, yielding the lemma statement.

Note that $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$ indeed covers all neighbors of u in $\mathcal{U}$ since $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$ covers all of $\mathcal{U}$. ◀

▶ **Lemma 22.** *Consider any $\mathcal{A}_i$. The only cop combinations not containing $\mathcal{A}_i$ that cover all neighbors of $\mathcal{A}_i$ in $\mathcal{U}$ are $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$ and $\{\mathcal{S}_0, \mathcal{T}_i, \mathcal{S}_2, \dots, \mathcal{S}_{k-1}\}$.*

Proof. Analogously to the proof of Lemma 7, using the subset $B = \{\mathcal{E}_{0,i}, \dots, \mathcal{E}_{2k-1,i}\}$ of the set of neighbors of $\mathcal{A}_i$ in $\mathcal{U}$, one can show that the only nodes $\ne \mathcal{A}_i$ that cover at least 2 nodes from B are $\mathcal{S}_0, \dots, \mathcal{S}_{2k-1}$ and $\mathcal{T}_i$. Moreover, these nodes cover each exactly 2 of the $2k$ nodes from B and the $\mathcal{S}$ nodes together cover all of B while T_i covers exactly the same nodes from B as $\mathcal{S}_1$. Thus, the only two cop combinations that cover all nodes in B are $\{\mathcal{S}_0, \dots, \mathcal{S}_{k-1}\}$ and $\{\mathcal{S}_0, \mathcal{T}_i, \mathcal{S}_2, \dots, \mathcal{S}_{k-1}\}$ and these two cop combinations indeed cover all nodes in B. The lemma statement follows. ◀

▶ **Lemma 23.** *Consider any $\mathcal{X}_i^j$. The only cop combination not containing $\mathcal{X}_i^j$ that covers all neighbors of $\mathcal{X}_i^j$ in $\mathcal{U}$ is $\{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$.*

Proof. Analogously to the proof of Lemma 8, this follows from the proof of Lemma 21. ◀

▶ **Lemma 24.** *Consider any $\mathcal{R}_i$. The only cop combinations not containing any $\mathcal{R}_j$ with $j \equiv i \pmod 3$ that cover all exits of $\mathcal{R}_i$ are $\{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$, $\{\mathcal{S}_0, \mathcal{T}_j, \mathcal{S}_2, \ldots, \mathcal{S}_{k-1}\}$ for any j, and $\{\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}}\}$ for any index tuple $(j_0, \ldots, j_{k-1})$ satisfying the following condition: For each $h \in \{0, \ldots, k-2\}$ it holds that*

1. *$j_h \pmod 3 = 0$ and $0 \le j_{h+1} \le \hat{n}/3 - 1$ or*
2. *$j_h \pmod 3 = 1$ and $\hat{n}/3 \le j_{h+1} \le 2\hat{n}/3 - 1$ or*
3. *$j_h \pmod 3 = 2$ and $2\hat{n}/3 \le j_{h+1} \le \hat{n} - 1$.*

Proof. Assume for this proof that no considered cop combination contains any $\mathcal{R}_j$ with $j \equiv i \pmod 3$. We call a node $\neq \mathcal{R}_i$ *base-covering* if it covers at least three nodes from $\left\{\mathcal{X}_0^{i \pmod 3}, \ldots, \mathcal{X}_3^{i \pmod 3}\right\}$, and *$q$-covering* if it covers $\mathcal{X}_{6q+5}^{i \pmod 3}$. Thus, any cop combination covering all exits of $\mathcal{R}_i$ has to contain a q-covering node for every $0 \le q \le k-2$.

Checking Tables 4, 5, 6, and the specification of the $\mathcal{X}$ nodes as a subset of $\mathcal{L} \setminus \mathcal{L}^*$,[10] we see that for any $0 \le q, q' \le k-2$ satisfying $q \neq q'$, the following holds: If a node is base-covering, then it is not q-covering; if a node is q-covering, then it is not base-covering; if a node is q-covering, then it is not q'-covering. Moreover, if a node is q-covering, then it covers at most one node from $\left\{\mathcal{X}_0^{i \pmod 3}, \ldots, \mathcal{X}_3^{i \pmod 3}\right\}$ if $q = 0$, and no node from $\left\{\mathcal{X}_0^{i \pmod 3}, \ldots, \mathcal{X}_3^{i \pmod 3}\right\}$ if $q \ge 1$.

Hence, it follows that any cop combination covering all exits of $\mathcal{R}_i$ has to contain exactly one q-covering node for every $0 \le q \le k-2$ and exactly one base-covering node; furthermore, these k nodes are pairwise different nodes. Again, checking Tables 4, 5 and 6, we see that each base-covering node and each $(k-2)$-covering node cover at most 3 exits of $\mathcal{R}_i$, whereas for $0 \le q \le k-3$, each q-covering node covers at most 6 exits of $\mathcal{R}_i$. Since $\mathcal{R}_i$ has $6(k-1)$ exits, any two different nodes in a cop combination that covers all exits of $\mathcal{R}_i$ have to cover disjoint sets of exits. Moreover, the base-covering node and the $(k-2)$-covering node indeed have to cover 3 exits of $\mathcal{R}_i$ and the other q-covering nodes indeed 6 exits of $\mathcal{R}_i$. This implies that the base-covering node has to be $\mathcal{S}_0$ or a node from $\mathcal{C}$, the 0-covering node has to be $\mathcal{S}_1$, a node from $\mathcal{T}$ or a node from $\mathcal{D}^0$, and, for any $1 \le q \le k-3$, the q-covering node has to be $\mathcal{S}_{q+1}$ or a node from $\mathcal{D}^q$. Furthermore, the $(k-2)$-covering node has to be $\mathcal{S}_{k-1}$, a node from $\mathcal{D}^{k-2}$, or an $\mathcal{E}$ node.[11]

As a special case, if $k = 2$, the 0-covering node v is also the $(k-2)$-covering node. In this case, node v has to be $\mathcal{S}_1$, a node from $\mathcal{T}$, or a node from $\mathcal{D}^0$ (note that v cannot be an $\mathcal{E}$ node in this case).

For the case of $k \ge 3$, we observe that no base-covering or q-covering node v, for $q \le k-4$, covers a node from $\left\{\mathcal{X}_{6(k-2)}^{i \pmod 3}, \ldots, \mathcal{X}_{6(k-1)-1}^{i \pmod 3}\right\}$ and that any $(k-3)$-covering node covers at most 3 of these nodes. Since any $\mathcal{E}$ node covers at most 1 of these nodes, it cannot be the case that the $(k-2)$-covering node is an $\mathcal{E}$ node.

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[10] Note that for any fixed $0 \le i \le 2$ any node from $\mathcal{E}$ covers at most 1 node from $\left\{\mathcal{X}_0^i, \ldots, \mathcal{X}_3^i\right\} \cup \left\{\mathcal{X}_{6q+5}^i \mid 0 \le q \le k-2\right\}$ by Lemma 19, since these $\mathcal{X}$ nodes are borrowed from a set of "parallel" lines from $\mathcal{L} \setminus \mathcal{L}^*$.

[11] No $\mathcal{E}$ node can be base-covering or q-covering for some $0 \le q \le k-3$ since any $\mathcal{E}$ node covers at most 1 node from $\left\{\mathcal{X}_0^i, \ldots, \mathcal{X}_3^i\right\}$ and at most 3 nodes from $\mathcal{X}^i$ for any $0 \le i \le 2$, by Lemma 19.

Let $V_k = \{v_0, \ldots, v_{k-1}\}$ be a cop combination that covers all exits of $\mathcal{R}_i$ and assume w.l.o.g. that v_0 is the base-covering node and that v_q is the $(q-1)$-covering node, for all $1 \leq q \leq k-1$. Consider the subset $W = \left\{\mathcal{X}_0^{i \pmod 3}, \ldots, \mathcal{X}_5^{i \pmod 3}\right\}$ of the set of exits of $\mathcal{R}_i$. Since no $S_{\geq 2}$ or $D^{\geq 1}$ node covers a node from W, we get that the only two nodes from V_k that can cover nodes from this subset are the base-covering node v_0 and the 0-covering node v_1. Thus, v_0 and v_1 have to cover complementary subsets of W and, checking Tables 4, 5 and 6, we see that this is only the case if $v_0 = \mathcal{S}_0$ and $v_1 = \mathcal{S}_1$, or $v_0 = \mathcal{S}_0$ and $v_1 \in \mathcal{T}$, or $v_0 = \mathcal{C}_{j_0}$ and $v_1 = \mathcal{D}^0_{j_1}$ such that one of the three properties from the lemma statement is satisfied for $h = 0$.

Using the next 6 exits $\mathcal{X}_6^{i \pmod 3}, \ldots, \mathcal{X}_{11}^{i \pmod 3}$, we obtain a similar statement about which combinations of nodes v_1 and v_2 can be, and so on. Combining the knowledge about any two consecutive nodes from our cop combination, we obtain exactly the lemma statement. ◀

Using, Lemma 24, we define exit-blocking, proper exit-blocking and forcing proper exit-blocking cop combinations analogously to the definitions in the 2-cop case. Again, a proper exit-blocking cop combination $\{\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}}\}$ is forcing if and only if $j_0 \in \{0, \hat{n}/3, 2\hat{n}/3\}$.

▶ **Lemma 25.** *Let* $(\{\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}}\} = \{c_0(t), \ldots, c_{k-1}(t)\}, \{c_0(t+1), \ldots, c_{k-1}(t+1)\}, \ldots, \{c_0(t+h), c_1(t+h)\} = \{\mathcal{C}_{j'_0}, \mathcal{D}^0_{j'_1}, \ldots, \mathcal{D}^{k-2}_{j'_{k-1}}\})$ *be a sequence of proper exit-blocking cop combinations describing the combined movement of the k cops from time t to time $t+h$. If* $\{\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}}\}$ *and* $\{\mathcal{C}_{j'_0}, \mathcal{D}^0_{j'_1}, \ldots, \mathcal{D}^{k-2}_{j'_{k-1}}\}$ *are forcing and* $j_0 \neq j'_0$*, then* $h \in \Omega(\hat{n})^k$.

Proof. Assume that $\{\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}}\}$ and $\{\mathcal{C}_{j'_0}, \mathcal{D}^0_{j'_1}, \ldots, \mathcal{D}^{k-2}_{j'_{k-1}}\})$ are forcing and $j_0 \neq j'_0$. Assume further w.l.o.g. that the cop staying in the $\mathcal{C}$ nodes is Cop 0 and, for any $0 \leq q \leq k-2$, the cop staying in the $\mathcal{D}^q$ nodes is Cop $q+1$.

Then, with a proof analogous to the proof of Lemma 10, we see that Cop 0 has to take at least $\hat{n}/3$ steps in the $\mathcal{C}$ cycle to get from $\mathcal{C}_{j_0}$ to $\mathcal{C}_{j'_0}$, Cop 1 has to take at least $\hat{n}/3 - 1$ steps between any two of those steps of Cop 0, Cop 2 has to take at least $\hat{n}/3 - 1$ steps between any two of those steps of Cop 1, and so on. We obtain $h \in \Omega(\hat{n})^k$. ◀

B.3 The Robber's Strategy

Here, we adapt the strategy for the robber from the 2-cop case to the case of k cops as follows:

If the cops are in $\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}$ in round 0, then the robber starts in $\mathcal{R}_0$, otherwise the robber starts in some node in $\mathcal{U}$ that is not covered by any of the cops (which exists by Lemma 21). Depending on where the cops are, the robber moves as specified in Table 7 (as long as she is not captured yet).

Analogously to the arguments provided in the 2-cop case, Lemmas 21, 22 and 23 imply that the specified strategy is well-defined and covers all possibly occurring situations.

B.4 The Cops' Strategy

Here, we adapt the cops' strategy to the case of k cops as follows:

Cop i starts in $\mathcal{S}_i$ in round 0. Depending on where the robber is, the cops move as specified in Table 3. Again, we have the exception that if a cop can capture the robber immediately, then she does so, overriding any possible instruction from the table.

By analogous arguments to the ones provided in the 2-cop case, we see that the specified strategy is well-defined and that no other situations than the specified ones can actually

$\{c_0(t), \ldots, c_{k-1}(t)\}$	$r(t-1)$	$r(t)$
$\neq \{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$	some node in $\mathcal{U}$	some uncovered node in $\mathcal{U}$
$\{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$	some node in $\mathcal{U}$	some node in $\mathcal{A}$
$\{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$ or $\{\mathcal{S}_0, \mathcal{T}_i, \mathcal{S}_2, \ldots, \mathcal{S}_{k-1}\}$	$\mathcal{A}_i$	$\mathcal{R}_0$
$\neq \{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$ and $\neq \{\mathcal{S}_0, \mathcal{T}_i, \mathcal{S}_2, \ldots, \mathcal{S}_{k-1}\}$	$\mathcal{A}_i$	some uncovered node in $\mathcal{U}$
not covering all exits of $\mathcal{R}_i$	$\mathcal{R}_i$	some uncovered exit of $\mathcal{R}_i$
covering all exits of $\mathcal{R}_i$	$\mathcal{R}_i$	the uncovered node from $\{\mathcal{R}_{i-1}, \mathcal{R}_i, \mathcal{R}_{i+1}\}$ with smallest absolute index; if all are covered, stay in $\mathcal{R}_i$
$\neq \{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$	$\mathcal{X}_i^j$	some uncovered node in $\mathcal{U}$
$\{\mathcal{S}_0, \ldots, \mathcal{S}_{k-1}\}$	$\mathcal{X}_i^j$	some node from $\{\mathcal{R}_{-1}, \mathcal{R}_0, \mathcal{R}_1\}$

Table 7 The robber's strategy

$r(t-1)$	$(c_0(t-1), \ldots, c_{k-1}(t-1))$	$(c_0(t), \ldots, c_{k-1}(t))$
$\mathcal{A}_i$	$(\mathcal{S}_0, \ldots, \mathcal{S}_{k-1})$	$(\mathcal{S}_0, \mathcal{T}_i, \mathcal{S}_2, \ldots, \mathcal{S}_{k-1})$
$\neq \mathcal{A}_i$ for all i	$(\mathcal{S}_0, \ldots, \mathcal{S}_{k-1})$	$(\mathcal{S}_0, \mathcal{T}_j, \mathcal{S}_2, \ldots, \mathcal{S}_{k-1})$ for some j
arbitrary	$(\mathcal{S}_0, \mathcal{T}_i, \mathcal{S}_2, \ldots, \mathcal{S}_{k-1})$	$\left(\mathcal{C}_0, \mathcal{D}_0^0, \ldots, \mathcal{D}_0^{k-2}\right)$
$\neq \mathcal{C}_h$ for all h	$\left(\mathcal{C}_{j_0}, \mathcal{D}_{j_1}^0, \ldots, \mathcal{D}_{j_{k-1}}^{k-2}\right)$	the first of the following cop combinations that covers all nodes in $\mathcal{X}$: $\left(\mathcal{C}_{j_0+1 \pmod{\hat{n}}}, \mathcal{D}^0_{j_1+1 \pmod{\hat{n}}}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}+1 \pmod{\hat{n}}}\right)$ $\left(\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1+1 \pmod{\hat{n}}}, \ldots, \mathcal{D}^{k-2}_{j_{k-1}+1 \pmod{\hat{n}}}\right)$ $\left(\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \mathcal{D}^1_{j_2+1 \pmod{\hat{n}}} \ldots, \mathcal{D}^{k-2}_{j_{k-1}+1 \pmod{\hat{n}}}\right)$ $\vdots$ $\left(\mathcal{C}_{j_0}, \mathcal{D}^0_{j_1}, \ldots, \mathcal{D}^{k-3}_{j_{k-2}}, \mathcal{D}^{k-2}_{j_{k-1}+1 \pmod{\hat{n}}}\right)$
$\mathcal{C}_h$	$\left(\mathcal{C}_{j_0}, \mathcal{D}_{j_1}^0, \ldots, \mathcal{D}_{j_{k-1}}^{k-2}\right)$	$(\mathcal{S}_0, \mathcal{P}, \text{arbitrary}, \ldots, \text{arbitrary})$

Table 8 The cops' strategy

occur. The only exception is that it is not a priori obvious that, in the case where the $r(t-1)$ column specifies "$\neq \mathcal{C}_h$ for all h", at least one of the presented output cop combinations actually covers all nodes in $\mathcal{X}$. This will be taken care of by Lemma 29.

B.5 A Lower Bound for the Robber's Strategy

Here, we show that k cops need time $(\Omega(\hat{n}))^{k+1}$ in order to capture the robber in $G_{\hat{n}}^k$ if the robber follows the strategy specified in Section B.3. For any constant k, this translates to a capture time of $\Omega(n^{k+1})$, and for non-constant $k = k(n)$, to a capture time of $(\Omega(n/k))^{k+1}$. The latter bound holds for all $k(n) \in o(\sqrt{n})$ and for $k(n) \in \Theta(\sqrt{n})$ with a small enough constant in the $\mathcal{O}$-notation.

Again, for convenience, we assume throughout the following lower bound considerations that if a cop can capture the robber immediately, then she does so.

▶ **Lemma 26.** *If the robber follows the strategy specified in Section B.3, then the following holds: If the robber is in some node $\mathcal{R}_i$ at time t and is not captured at time $t+1$, then none of the k cops can be in some node $\mathcal{R}_j$ with $j \equiv i \pmod 3$ at time $t+1$.*

Proof. This follows by a proof completely analogous to the proof of Lemma 11. ◀

▶ **Lemma 27.** *If the robber follows the strategy specified in Section B.3, then she can only be captured in $\mathcal{R}_{\hat{n}}$ or $\mathcal{R}_{-\hat{n}}$.*

Proof. This follows by a proof completely analogous to the proof of Lemma 12. ◀

▶ **Lemma 28.** *If the robber follows the strategy specified in Section B.3, then k cops need time $\Omega(\hat{n})^{k+1}$ to capture the robber in $G_{\hat{n}}^k$.*

Proof. This follows by a proof completely analogous to the proof of Lemma 13. ◀

B.6 An Upper Bound for the Cops' Strategy

Here, we show a matching upper bound to the lower bound presented in the previous section, for the graphs $G_{\hat{n}}^k$. The cops achieve this upper bound by following the strategy presented in Section B.4 which shows that the specified strategies for the robber and for the cops are asymptotically optimal in the graph class $\{G_{\hat{n}}^k\}$.

▶ **Lemma 29.** *If $(\mathcal{C}_{j_0}, \mathcal{D}_{j_1}^0, \ldots, \mathcal{D}_{j_{k-1}}^{k-2})$ is an exit-blocking cop combination, then at least one of the output cop combinations in Table 8, in the row where $r(t-1)$ is specified as "$\neq \mathcal{C}_h$ for all h", is an exit-blocking cop combination.*

Proof. Let $(\mathcal{C}_{j_0}, \mathcal{D}_{j_1}^0, \ldots, \mathcal{D}_{j_{k-1}}^{k-2})$ be an exit-blocking cop combination and assume w.l.o.g. that Cop 0 is in $\mathcal{C}_{j_0}$, Cop 1 is in $\mathcal{D}_{j_1}^0$, and so on. Then, by applying the arguments from the proof of Lemma 14 to the last 6 exits $\{\mathcal{X}_{6(k-2)}^h, \ldots, \mathcal{X}_{6(k-2)+5}^h\}$ for any $0 \leq h \leq 2$, we observe the following: If Cop $k-1$ stays in $\mathcal{D}_{j_{k-1}}^{k-2}$, then Cop $k-2$ can make sure that those 6 exits are covered by also staying in $\mathcal{D}_{j_{k-2}}^{k-3}$. If Cop $k-1$ moves instead to $\mathcal{D}_{j_{k-1}+1 \pmod{\hat{n}}}^{k-2}$, then Cop $k-2$ can make sure that those 6 exits are covered by also staying in $\mathcal{D}_{j_{k-2}}^{k-3}$ or by moving to $\mathcal{D}_{j_{k-2}+1 \pmod{\hat{n}}}^{k-3}$, depending on if j_{k-1} is not or is in $\{\hat{n}/3-1, 2\hat{n}/3-1, \hat{n}-1\}$.

In a similar fashion, Cop $k-3$ can ensure that the 6 previous exits $\{\mathcal{X}_{6(k-3)}^h, \ldots, \mathcal{X}_{6(k-3)+5}^h\}$ are covered, by staying in her node or moving to the next node in $\mathcal{D}^{k-4}$, depending on if Cop $k-2$ stays or moves. By inductively applying this argument, we see that all exits can be covered by a combined step of the cops where each cop either stays in her node or moves to the next node in her $\mathcal{C}$ or $\mathcal{D}^q$ cycle. Moreover, if Cop q stays in her node during this combined step, then all cops q' with $q' < q$ also stay in their nodes. It follows that at least one of the cop combinations specified in the lemma is an exit-blocking cop combination. ◀

▶ **Lemma 30.** *Let $r(t) \in \mathcal{R}$ and $(c_0(t+1), \ldots, c_{k-1}(t+1)) = (\mathcal{C}_0, \mathcal{D}_0^0, \ldots, \mathcal{D}_0^{k-2})$ for some point in time t. If the robber leaves $\mathcal{R}$ at some later point in time t', i.e., if $r(t') \notin \mathcal{R}$ for some $t' > t$, then the robber will be captured at time $t'' \leq t' + 2$, provided the k cops follow the strategy specified in Section B.4.*

Proof. This follows by a proof completely analogous to the proof of Lemma 15. ◀

▶ **Lemma 31.** *If the k cops follow the strategy specified in Section B.4, then they capture the robber in time $(\mathcal{O}(\hat{n}))^{k+1}$ in $G_{\hat{n}}^k$.*

Proof. Analogously to the proof of Lemma 16, one can show that the robber is forced to the $\mathcal{R}$ path in at most 2 moves and that we can assume that she stays there forever. Also analogously, one can obtain the following picture for the combined movement of the cops (after they reach the cop combination $\{\mathcal{C}_0, \mathcal{D}_0^0, \ldots, \mathcal{D}_0^{k-2}\}$ where we assume w.l.o.g. that Cop 0 is in $\mathcal{C}_0$, Cop 1 is in $\mathcal{D}_0^0$, and so on): Cop $k-1$ continuously moves in $\mathcal{D}^{k-2}$ in a clockwise or counterclockwise fashion, Cop $k-2$ takes one step for any $\hat{n}/3$ steps of Cop $k-1$, Cop $k-3$ takes one step for any $\hat{n}/3$ steps of Cop $k-2$, and so on, up to Cop 0. Moreover, whenever Cop 0 reaches a node from $\{\mathcal{C}_0, \mathcal{C}_{\hat{n}/3}, \mathcal{C}_{2\hat{n}/3}\}$ (which happens at least once in any consecutive $(\hat{n}/3)^k \in (\mathcal{O}(\hat{n}))^k$ time steps) the number of $\mathcal{R}$ nodes the robber can be in without being captured immediately decreases by 1. As there are only $\mathcal{O}(\hat{n})$ $\mathcal{R}$ nodes in total, the robber will be captured in time $(\mathcal{O}(\hat{n}))^{k+1}$. ◀

Now, finally, Lemmas 28 and 31 together yield Theorem 1 with the same (short) line of argumentation as in the case of $k = 2$.