Distributed Algorithms

- Message Passing
- Shared Memory
Example: Maximal Independent Set (MIS)

- Given a network with $n$ nodes, nodes have unique IDs.
- Find a Maximal Independent Set (MIS) – a non-extendable set of pair-wise non-adjacent nodes.
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Traditional (sequential) computation:
The simple greedy algorithm finds MIS (in linear time)
What about a Distributed Algorithm?

- Nodes are agents with unique ID’s that can communicate with neighbors by **sending messages**. In each **synchronous round**, every node can send a (different) message to each neighbor.
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```
17
/   \
/     \
69    11

10 --- 7
```

Each round:
1. send msgs
2. rcv msgs
3. compute
A Simple Distributed Algorithm

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS → join MIS
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What’s the problem with this distributed algorithm?

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```
69   17   11   10   7   4   3   1
```

- What if we have minor changes?

```
69   17   11   10   7   4   3   1
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Example

- Wait until all neighbors with higher ID decided
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• What if we have minor changes?

![Network diagram with nodes 69, 17, 11, 10, 7, 4, 3, 1]
Example

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS $\implies$ join MIS

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![Diagram](image)

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![Diagram showing nodes with IDs 69, 17, 11, 10, 7, 4, 3, 1]

- What if we have minor changes?

![Diagram showing nodes with ID 69, 17, 11, 10, 7, 4, 3, 1 and butterfly]

- Proof by animation: In the worst case, the algorithm is slow (linear in the number of nodes). In addition, we have a terrible "butterfly effect".
What about a Fast Distributed Algorithm?

- Can you find a distributed algorithm that is polylogarithmic in the number of nodes $n$, for any graph?
What about a Fast Distributed Algorithm?

• Surprisingly, for deterministic distributed algorithms, this is an open problem!

• However, randomization helps! In each synchronous round, nodes should choose a random value. If your value is larger than the value of your neighbors, join MIS!
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- How many synchronous rounds does this take in expectation (or whp)?
Analysis

- Event \((u \to v)\): node \(u\) got largest random value in combined neighborhood \(N_u \cup N_v\).
- We only count edges of \(v\) as deleted.

- Similarly event \((v \to u)\) deletes edges of \(u\).
- We only double-counted edges.
- Using linearity of expectation, in expectation at least half of the edges are removed in each round.
- In other words, whp it takes \(O(\log n)\) rounds to compute an MIS.
Results: MIS

- General Graphs, Randomized [Alon, Babai, and Itai, 1986]
  [Israeli and Itai, 1986]
  [Luby, 1986]
  [Métivier et al., 2009]

- Decomposition, Determ. [Awerbuch et al., 1989]
  [Panconesi et al., 1996]

- Naïve Algo
Local Algorithms

- Each node can exchange a message with all neighbors, for $t$ communication rounds, and must then decide.
- Or: Given a graph, each node must determine its decision as a function of the information available within radius $t$ of the node.
- Or: Change can only affect nodes up to distance $t$.
- Or: ...
Locality

Local Algorithms

Sublinear Algorithms
Locality is Everywhere!

Self-Assembling Robots

Applications e.g. Multicore

Self-Stabilization

Local Algorithms

Dynamics

Sublinear Algorithms
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- Dynamics
- Sublinear Algorithms
Afek, Alon, Barad, et al., 2011
What about an **Even Faster** Distributed Algorithm?

- Since the 1980s, nobody was able to improve this simple algorithm.

- **What about** lower bounds?

- There is an interesting lower bound, essentially using a Ramsey theory argument, that proves that an MIS needs at least $\Omega(\log^* n)$ time.
  - $\log^*$ is the so-called iterated logarithm – how often you need to take the logarithm until you end up with a value smaller than 1.
  - This lower bound already works on simple networks such as the linked list
Coloring Lower Bound on Oriented Ring

• Build graph $G_t$, where nodes are possible views of nodes for distributed algorithms of time $t$. Connect views that could be neighbors in ring.

• Here is for instance of $G_1$:

• Chromatic number of $G_t$ is exactly minimum possible colors in time $t$. 
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1 \log^* n \quad \log n \quad n^\epsilon \quad n

Linked List [Linial, 1992]

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Results: MIS

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Results: MIS

\[ |IS(N_2)| \in O(1) \]

Growth-Bounded Graphs
[Schneider et al., 2008]

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- Other problems e.g., [Kuhn et al., 2006]
- e.g., coloring, CDS, matching, max-min LPs, facility location
- e.g., covering/packing LPs with only local constraints: constant approximation in time \( O(\log n) \) or \( O(\log^2 \Delta) \)
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Example: Minimum Vertex Cover (MVC)

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Differences between MIS and MVC

- Central (non-local) algorithms: MIS is trivial, whereas MVC is NP-hard
- Instead: Find an MVC that is “close” to minimum (approximation)
- Trade-off between time complexity and approximation ratio

MVC: Various simple (non-distributed) 2-approximations exist!
- What about distributed algorithms?!!
Finding the MVC (by Distributed Algorithm)

- Given the following bipartite graph with $|S_0| = \delta |S_1|$
- The MVC is just all the nodes in $S_1$
- Distributed Algorithm...
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- Distributed Algorithm...
$N_2$ (node in $S_0$) 

$N_2$ (node in $S_1$)
Graph is “symmetric”, yet highly non-regular!
Lower Bound: The Argument

- The example graph is for $t = 3$.
- All edges are in fact special bipartite graphs with large enough girth.

- If you use the graph of recursion level $t$, then a distributed algorithm cannot find a good MVC approximation in time $t$. 
• Choose degrees $\delta_i$ such that $\delta_{i+1}/\delta_i = 2^i \delta$.
• We have $|S_0| > \delta/2 \cdot |L_1|$, with $|L_1|$ nodes on level 1
Lower Bound: The Math

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Now $\delta, n, \Delta$ are depending on the recursion level $t$. 
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- Now $\delta, n, \Delta$ are depending on the recursion level $t$.

Graph useful for proving lower bounds in sublinear algs?
Lower Bound: Results

- We can show that for $\epsilon > 0$, in $t$ time, the approximation ratio is at least
  \[ \Omega \left( n^{\frac{1}{4} - \epsilon} \right) \text{ and } \Omega \left( \Delta^{\frac{1}{t + 1}} \right) \]

- Constant approximation needs at least $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ time.
- Polylog approximation $\Omega(\log \Delta / \log \log \Delta)$ and $\Omega(\sqrt{\log n / \log \log n})$. 
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 Tight for MVC
Many “local looking” problems need non-trivial $t$, in other words, the bounds $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ hold for a variety of classic problems.
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Naïve Algo

e.g., covering/packing
LPs with only local
constraints: constant
approximation in time
O(log n) or O(log² Δ)

Linked List
[Linial, 1992]

General Graphs
[Kuhn et al., 2004, 2006]

Open

1
log* n
√log n ... log n
n^ε
n
Summary

1. \( \log^* n \) 
   - Growth-Bounded Graphs (various problems)
     - E.g., dominating set approximation in planar graphs

2. \( \sqrt{\log n} \ldots \log n \) 
   - Approximations of dominating set, vertex cover, etc.

3. Diameter 
   - MIS, maximal matching, etc.
   - MST, Sum, etc.

Covering and packing LPs

E.g., dominating set approximation in planar graphs
Thank You!

Questions & Comments?

Thanks to my co-authors
Fabian Kuhn
Thomas Moscibroda
Johannes Schneider

www.disco.ethz.ch
Open Problems

• Close the gap between $\sqrt{\log n}$ and $\log n$ (for randomized algorithms)!
• Find a fast deterministic MIS algorithm (or strong det. lower bound)!
• Where are the boundaries between constant, log*, log, and diameter?
• What about algorithms that cannot even exchange messages?
• Can the lower bound graph be used in the context of sublinear algorithms?